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YANG-MILLS FOR NON-COMMUTATIVE TWO-TORI

Dedicated to

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in Celebration of their Sixtieth Birthdays

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The first author initiated in [C1] the subject of differential geometry for projective modules ("vector bundles") over C*-algebras, with its apparatus of connections, curvature and Chern classes. The principal examples discussed in [C1] were the non-commutative tori and the projective modules over them.

In the present paper we take the first small steps in extending Yang-Mills theory to projective modules over a C*-algebra. Our motivation for doing that can be explained as follows. A non-commutative C*-algebra such as the irrational rotation C*-algebra A_θ describes a non-commutative analogue of an ordinary 2-torus, with smooth structure prescribed by the dense subalgebra A_θ^∞ (cf. [C1]) of smooth elements. The operation which allows one to pass from the ordinary torus $\mathbb{T}^2 = V$ to the pseudo torus, or more precisely from the algebra $C^\infty(V)$ to the algebra A_θ^∞, is the introduction of new phase factors in the product rule. It is however not clear at all how to associate to a non-commutative algebra such as A_θ^∞ an ordinary manifold which would be its

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"manifold shadow". In particular, ideas related to the spectrum cannot work, since for $\theta \notin \mathbb{Q}$, A_θ^∞ is a simple algebra so that its spectrum is a point. This goal is exactly what the Yang-Mills problem achieves. More specifically, we shall prove in this paper that if Ξ is a finite projective module over A_θ^∞ which is not a multiple of any other projective module, then the moduli space for connections on Ξ^d which minimize the Yang-Mills functional is homeomorphic to $(\mathbb{T}^2)^d/\Sigma_d$ where $\mathbb{T}^2$ is the ordinary 2-torus and Σ_d is the group of permutations of d objects acting by permuting the components of $(\mathbb{T}^2)^d$. Any finite projective module over A_θ^∞ is of the above form Ξ^d.

Projective modules over higher-dimensional non-commutative tori are studied in [R5,R6], and in particular, many projective modules which admit connections with constant curvature are constructed. Most of the results discussed in the present paper extend to these modules. However, there are other projective modules which do not admit connections of constant curvature, and for which the determination of the moduli space will be more difficult.

Many other directions for the extension of Yang-Mills theory are suggested by, among many possibilities, the expositions in [AB,FU].

1. THE YANG-MILLS FUNCTIONAL

We work in the setting introduced in [C1] in which the C^∞ structure is defined by an action of a Lie group, rather than the more general setting developed in [C4]. Thus we let G be a connected Lie group with Lie algebra L, and we let (A,G,α) be a C^*-dynamical system. We let A^∞ denote the dense $*$-subalgebra of A consisting of the C^∞-vectors for the action α. (See the appendix of [C2].) Then the infinitesimal form of α gives an action δ of L as a Lie algebra of derivations on A^∞.

We will assume that A is unital. Then finitely generated projective (right) A-modules are the appropriate generalization of complex vector bundles over a compact space. For brevity we will from now on say "projective" when we mean "finitely generated projective". As indicated in Lemma 1 of [C1], every projective A-module, Ξ, has a C^∞ version, that is, there is a projective A^∞-module Ξ^∞ such that Ξ is isomorphic to $\Xi^\infty \otimes_{A^\infty} A$. Since we will never work with A and Ξ, but only with A^∞ and Ξ^∞, we will for notational simplicity denote the latter by A and Ξ from now on.

As discussed in [C1], we can always equip Ξ with a Hermitian metric, that is, an A-valued positive-definite inner-product $< \, , \, >_A$ such that

$$<\xi,\eta>_A^* = <\eta,\xi>_A, \quad <\xi,\eta a>_A = <\xi,\eta>_A a$$

for ξ, $\eta \in \Xi$ and $a \in A$, for which Ξ is self-dual. We will always assume that Ξ has been so equipped.

Yang-Mills theory is concerned with the set of connections (i.e. gauge potentials) on a vector bundle. In our setting [C1], a connection is a linear map ∇ from Ξ to $\Xi \otimes L^*$ such that

$$\nabla_X(\xi a) = (\nabla_X(\xi))a + \xi(\delta_X(a))$$

for all $X \in L$, $\xi \in \Xi$ and $a \in A$. Furthermore, the connections are required to be compatible with the Hermitian metric, that is,

$$\delta_X(<\xi,\eta>_A) = <\nabla_X\xi,\eta>_A + <\xi,\nabla_X\eta>_A$$

for all $X \in L$, ξ, $\eta \in \Xi$. As discussed in [C1], such compatible connections always exist. We will denote the set of compatible connections by $CC(\Xi)$. If ∇^0 and ∇ are any two connections, then $\nabla_X - \nabla_X^0$ is an element of $E = End_A(\Xi)$, for each $X \in L$. If ∇ and ∇^0 are both compatible with the Hermitian metric, then $\nabla_X - \nabla_X^0$ is a skew-adjoint element of E for each $X \in L$. (Note that E is a pre-C*-algebra.) Thus, once we have fixed a compatible connection ∇, every other compatible connection is of the form $\nabla + \mu$ where μ is a linear map from L into E_s, the set of skew-adjoint elements of E. In other words, $CC(\Xi)$ is an affine space with vector space consisting of the linear maps from L to E_s.

The curvature (i.e. gauge field) of a connection ∇ is defined to be the alternating bilinear form Θ_∇ on L which measures the extent to which ∇ fails to be a Lie algebra homomorphism, that is,

$$\Theta_\nabla(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$$

for $X,Y \in L$. One finds that its values are in E. If ∇ is compatible with the Hermitian metric, then the values of Θ are in E_s.

The Yang-Mills functional measures the "strength" of the curvature of a connection. To define it we need some extra structure. Since L is playing the role of the tangent space of A, the analogue of a Riemannian metric on a manifold will be just an ordinary positive definite inner-product on L. We assume given such a Riemannian metric, which will remain fixed throughout. This Riemannian metric determines a bilinear form on the various spaces of alternating multilinear forms on L. If these forms have values in an algebra, such as E, then so will the corresponding bilinear form. We need the bilinear form especially on the space of alternating 2-forms with values in E. For computational purposes the easiest way to define it is in terms of an orthonormal basis, say $Z_1, \ldots, Z_n$, for L. Then given alternating E-valued 2-forms Φ and Ψ we let

$$\{\Phi,\Psi\}_E = \sum_{i<j} \Phi(Z_i \wedge Z_j)\Psi(Z_i \wedge Z_j),$$

which is an element of E.

We need next the analogue of integration over a manifold, and we need this to be G-invariant. Accordingly, we assume given a faithful trace, τ, on A, which is invariant under the action of L on A, that is, δ-invariant, so that $\tau(\delta_X(a)) = 0$ for all $X \in L$ and $a \in A$. Then τ determines a canonical faithful trace, τ_E, on E, as explained in Proposition 2.2 of [R3]. To define τ_E, we recall that on Ξ one defines an E-valued inner product, $<\ ,\ >_E$, by

$$<\xi,\eta>_E\zeta = \xi<\eta,\zeta>_A$$

Because Ξ is (finitely generated) projective, every element of E will be a finite linear combination of terms of form $<\xi,\eta>_E$. Then τ_E is determined by

$$\tau_E(<\xi,\eta>_E) = \tau(<\eta,\xi>_A).$$

The Yang-Mills functional, YM, is defined on CC(Ξ) by

$$YM(\nabla) = -\tau_E(\{\Theta_\nabla, \Theta_\nabla\}).$$

Since the values of Θ_∇ are in E_s, it is clear that YM(∇) is a non-negative real number. The Yang-Mills problem is that of determining the nature of the set of connections where YM attains its minimum, or, more generally, of the set of critical points for YM. The Yang-Mills equations are the Euler-Lagrange equations for the variational problem of finding the critical points of YM on CC(Ξ). Since we will not explicitly need these equations for our immediate purposes, we will not derive them here.

One can ask how YM depends on the choice of Hermitian metric on Ξ. Now if $[\ ,\ ]_A$ is another Hermitian metric, then it is easy to see that there is a positive invertible element e of E such that

$$[\xi,\eta]_A = <e\xi,\eta>_A$$

for all $\xi,\eta \in \Xi$. Suppose that ∇ is a connection on Ξ which is compatible with $[\ ,\ ]_A$. Then it is easily seen that $e^{1/2}\nabla e^{-1/2}$ is compatible with $<\ ,\ >_A$, and that if Θ' denotes the curvature of this latter connection, while Θ denotes the curvature of ∇, then $\Theta' = e^{1/2}\Theta e^{-1/2}$ so that $\tau(\Theta'(X,Y)^2) = \tau(\Theta(X,Y)^2)$. It follows that $YM(e^{1/2}\nabla e^{-1/2}) = YM(\nabla)$. (Note that $e^{-1/2}$ exists in E because A is closed under the holomorphic functional calculus as discussed in the appendix to [C2], so that E is also, since it can be realized as $pM_n(A)p$ for some projection p in some $M_n(A)$).

The Yang-Mills functional is invariant under a large group of symmetries, and consequently its set of minima (and critical points) will be also. This group is called the group of gauge transformations, or the gauge group of the second kind. In the present context we do not have an analogue of the gauge group of the first kind (the Lie group which is the structure group for the vector bundle), and so for brevity we will refer to the group of gauge transformations as simply the gauge group, as is frequently done anyway.

In our context, the gauge group is just the group, UE, of unitary elements of E, acting on $CC(\Xi)$ by conjugation. To be specific, for $u \in UE$, $\nabla \in CC(\Xi)$, we define $\gamma_u(\nabla)$ by

$$(\gamma_u(\nabla))_X \xi = u(\nabla_X(u^*\xi))$$

for $\xi \in \Xi$ and $X \in L$. It is easily verified that $\gamma_u(\nabla) \in CC(\Xi)$. It is also easily verified that

$$\Theta_{\gamma_u(\nabla)}(X,Y) = u\Theta_\nabla(X,Y)u^*$$

for $X,Y \in L$, and that

$$\{\Theta_{\gamma_u(\nabla)}, \Theta_{\gamma_u(\nabla)}\} = u\{\Theta_\nabla, \Theta_\nabla\}u^*.$$

In view of the trace used in the definition of YM, it follows that

$$YM(\gamma_u(\nabla)) = YM(\nabla)$$

for every $u \in UE$ and $\nabla \in CC(\Xi)$. Thus YM is a well-defined functional on the quotient space $CC(\Xi)/UE$. Consequently it is more appropriate to try to describe the set of minima for YM on this quotient space. Equivalently, if $MC(\Xi)$ denotes the set of compatible connections where YM attains its minimum, we wish to describe $MC(\Xi)/UE$. This quotient is called the moduli space for Ξ.

2. CONNECTIONS WITH CONSTANT CURVATURE

In the next section we will consider the specific case of the non-commutative two-tori, and the projective modules over them constructed in Theorem 7 of [C1] or Section 13 of [C3]. These modules, as well as many of the modules over higher-dimensional non-commutative tori constructed in [R5,R6], have very special properties. To begin with, the Lie algebra L involved is Abelian. Even more, the modules admit compatible connections with constant curvature, where for $\nabla \in CC(\Xi)$ we say that ∇ has *constant curvature* if there is a complex-valued alternating 2-form κ on L such that

$$\Theta_\nabla(X,Y) = \kappa(X,Y)I$$

for $X,Y \in L$, where I denotes the identity element of $E = \mathrm{End}_A(\Xi)$. (In fact, the values of κ are pure-imaginary, since the values of Θ_∇ are in E_s.) This simplifies enormously the analysis of the Yang-Mills question, since we have:

2.1. THEOREM. *Assume that* L *is Abelian, and that* Ξ *admits compatible connections with constant curvature. Then the set* $MC(\Xi)$ *of compatible connections where* YM *attains its minimum, consists exactly of all compatible connections with constant curvature. Furthermore, the curvatures of all these minimizing connections will be the same.*

PROOF. Let ∇^0 be a compatible connection with constant curvature $\Theta^0 = \kappa I$. As indicated earlier, any ∇ in $CC(\Xi)$ is then of the form $\nabla^0 + \mu$ where $\mu_X^* = -\mu_X$ for $X \in L$. In view of the fact that L is Abelian, the curvature, Θ, of ∇ is easily seen to be

$$\Theta = \Theta^0 + \Psi ,$$

where

$$\Psi(X,Y) = [\nabla_X,\mu_Y] - [\nabla_Y,\mu_X] + [\mu_X,\mu_Y].$$

Now for any $X \in L$, define the "covariant derivative" $\hat{\delta}_X$ on E by

$$\hat{\delta}_X(T) = [\nabla_X,T]$$

for $T \in E$. Then regardless of whether L is Abelian or whether ∇ has constant curvature, $\hat{\delta}_X$ is easily seen to be a derivation of E for each $X \in L$. (But $X \to \hat{\delta}_X$ is not in general a Lie algebra homomorphism).

2.2. LEMMA. *As earlier, let* τ *be a* δ*-invariant trace on* A, *and let* τ_E *be the corresponding trace on* E. *Let* $\nabla \in CC(\Xi)$, *with* $\hat{\delta}$ *its covariant derivative on* E. *Then* τ_E *is* $\hat{\delta}$*-invariant, in the sense that*

$$\tau_E(\hat{\delta}_X(T)) = 0$$

for all $T \in E$ *and* $X \in L$.

PROOF. Because ∇ is compatible, it is easily verified to be compatible for $\hat{\delta}$ and $<\ ,\ >_E$, in the sense that

$$\hat{\delta}_X(<\xi,\eta>_E) = <\nabla_X\xi,\eta>_E + <\xi,\nabla_X\eta>_E$$

for $\xi,\eta \in \Xi$ and $X \in L$. Then

$$\begin{aligned}\tau_E(\hat{\delta}_X(<\xi,\eta>_E)) &= \tau_E(<\nabla_X\xi,\eta>_E) + \tau_E(<\xi,\nabla_X\eta>_E)\\ &= \tau(<\eta,\nabla_X\xi>_A) + \tau(<\nabla_X\eta,\xi>_A)\\ &= \tau(\delta_X(<\eta,\xi>_A)) = 0.\end{aligned}$$

Since the operators of form $<\xi,\eta>_E$ span E, the desired conclusion follows. Q.E.D.

We return to the proof of Theorem 2.1. We see that

$$\Psi(X,Y) = \hat{\delta}_X(\mu_Y) - \hat{\delta}_Y(\mu_X) + [\mu_X,\mu_Y],$$

and so

$$\tau_E(\Psi(X,Y)) = 0$$

by Lemma 2.2. But then

$$\tau_E(\Theta^0(X,Y)\Psi(X,Y)) = \kappa(X,Y)\tau_E(\Psi(X,Y)) = 0$$

and similarly for Θ^0 and Ψ interchanged. It follows that

$$\tau_E(\{\Theta^0,\Psi\}) = 0 = \tau_E(\{\Psi,\Theta^0\}).$$

Consequently

$$YM(\nabla) = YM(\nabla^0) - \tau_E(\{\Psi,\Psi\}).$$

Now the values of Ψ are in E_s, and so the term $-\tau_E(\{\Psi,\Psi\})$ is non-negative. It follows that YM attains its minimum at ∇^0.

If YM is to attain its minimum also at ∇, then we must have $\tau_E(\{\Psi,\Psi\}) = 0$. Now we have assumed that τ is faithful. Then τ_E will also be faithful, for if $\tau_E(ee^*) = 0$, then for every $\xi \in \Xi$ we have

$$0 = \tau_E(e<\xi,\xi>_E e^*) = \tau_E(<e\xi,e\xi>_E) = \tau(<e\xi,e\xi>_A)$$

so that $e\xi = 0$ for all ξ, and so $e = 0$. Since Ψ has values in E_s, it follows that $\Psi = 0$. Thus

$$\Theta_\nabla = \Theta^0 = \kappa I.$$

Q.E.D.

3. HEISENBERG MODULES FOR NON-COMMUTATIVE TWO-TORI

We now study the specific modules over non-commutative two-tori which are constructed in Theorem 7 of [C1] and Section 13 of [C3]. (Their construction is generalized to higher-dimensional tori in [R5, R6]). Our notation will usually agree with that in [C1]. We recall that a non-commutative two-torus, A_θ, is specified by a real number θ (which need not be irrational). Let $\lambda = e(\theta)$, where here and later we let e denote the function defined by $e(t) = \exp(2\pi it)$ for real t. Then A_θ is the universal C*-algebra generated by two unitary operators, U_1 and U_2, subject to the condition that $U_2U_1 = \lambda U_1U_2$. Or rather, for us it is the C^∞ part thereof, with respect to the (dual) action of the torus group $G = T^2$. Thus, as discussed in [C1, C3] the elements of A_θ are of the form $\Sigma f(m,n)U_1^mU_2^n$ where f is a complex-valued Schwartz function on

$\mathbb{Z}^2$, that is, f vanishes rapidly at infinity. The Lie algebra L of G is two-dimensional, and we take as a basis for L the derivations δ_1 and δ_2 of A_θ defined by

$$\delta_k(U_k) = 2\pi i U_k, \quad \delta_k(U_j) = 0, \quad j \neq k.$$

Accordingly, to define connections we only need to specify them on this basis, and we write ∇_1 and ∇_2 for the corresponding operators.

Let $S(\mathbb{R})$ denote the usual Schwartz space of complex-valued functions on the real line. To specify a module Ξ we will assume given two integers, p and q with $q > 0$. Contrary to [C1] it will be convenient for us to assume that p and q are relatively prime, or that $p = 0$ and $q = 1$. Let K be a finite-dimensional Hilbert space, and let w_1 and w_2 be unitary operators on K such that

$$w_2 w_1 = \bar{e}(p/q) w_1 w_2$$

(where $\bar{e}$ is the complex-conjugate of e). We do not require that K be of dimension q, and consequently we recover the generality of [C1]. Contrary to [C1], we will for convenience assume that $w_1^q = I_K = w_2^q$. This entails no loss of generality, as w_1 and w_2 can always be adjusted to satisfy this relation. (The argument, in a related context, will be given near the end of this section.) Thus w_1 and w_2 provide a representation of the Heisenberg commutation relations for the finite group $\mathbb{Z}_q$. Since the irreducible such representation is unique up to equivalence, K decomposes into a direct sum of equivalent irreducible representations.

Let $\epsilon = (p/q) - \theta$, and let operators V_1 and V_2 be defined on $S(\mathbb{R})$ by

$$(V_1\xi)(s) = \xi(s - \epsilon), \quad (V_2\xi)(s) = e(s)\xi(s)$$

for $s \in \mathbb{R}$ and $\xi \in S(\mathbb{R})$. This makes evident the further requirement which we must put on p and q, when θ is rational, namely that $\epsilon \neq 0$. One finds that

$$V_2 V_1 = e(\epsilon) V_1 V_2 .$$

It follows that

$$(V_2 \otimes w_2)(V_1^* \otimes w_1) = \bar{\lambda}(V_1 \otimes w_1)(V_2 \otimes w_2).$$

We now let $\Xi = S(\mathbb{R}) \otimes K$, and let U_1 and U_2 be the operators acting on the *right* on Ξ defined by $U_i = V_i \otimes w_i$ for $i = 1,2$. Because U_1 and U_2 act on the right, we have $U_2U_1 = \lambda U_1U_2$. We view Ξ as consisting of K-valued Schwartz functions on $\mathbb{R}$. Then we define a Hermitian metric (that is, A-valued inner-product) on Ξ by

$$<\xi,\eta>_A(m,n) = \int_{-\infty}^{\infty} <w_2^n w_1^m \xi(s - m\epsilon),\eta(s)>\bar{e}(ns)ds,$$

where we assume, here and later, that the ordinary inner-product on K is linear in the *second* variable. Here we are also identifying a Schwartz function f on $\mathbb{Z}^2$ with the element $\Sigma f(m,n)U_1^m U_2^n$ of A $= A_\theta$. It is not difficult to verify that $<\xi,\eta>_A$ as defined above is indeed a Schwartz function on $\mathbb{Z}^2$, and that $< , >_A$ does define an A_θ-valued inner-product on Ξ. This can be used to show that Ξ is indeed a projective A_θ-module. (Modulo a Fourier transform and some changes of notation, the proofs are indicated in [R4].) If θ is irrational, then we obtain in this way all projective A_θ-modules up to isomorphism, except the free modules, as can be seen from the main theorem of [R4]. We will call the modules Ξ constructed above *Heisenberg* modules for A_θ, for reasons which will soon be apparent.

We now define a connection, ∇, on Ξ by

$$(\nabla_1\xi)(s) = 2\pi i(s/\epsilon)\xi(s), \quad (\nabla_2\xi)(s) = (d\xi/ds)(s).$$

(Note that these are reversed in [C1].) Straightforward calculations show that this is indeed a connection, and that it is compatible with the Hermitian metric defined above. Furthermore, a trivial calculation shows that the curvature of ∇, which is determined by $[\nabla_1,\nabla_2]$, is given by

$$[\nabla_1,\nabla_2] = -(2\pi i/\epsilon)I.$$

Thus ∇ has constant curvature. From Theorem 2.1 it follows that ∇ minimizes the Yang-Mills functional, and that every element of $MC(\Xi)$ must have curvature equal to that of ∇.

We wish to determine the moduli space $MC(\Xi)/UE$. This is facilitated by the fact that ∇ essentially defines a representation of the Heisenberg commutation relations, and this representation is the infinitesimal form of an evident representation of the (three-dimensional) Heisenberg Lie group, H. We view H as $\mathbb{R}^3$ with product defined by

$$(r,s,t)(r',s',t') = (r+r',s+s',t+t'+sr').$$

We let π denote the representation on $L^2(\mathbb{R},K)$ for which ∇_1 is the infinitesimal generator of the one-parameter group $(r,0,0)$ while ∇_2 is the generator for $(0,s,0)$. Thus

$$(\pi(r,0,0)\xi)(t) = e(rt/\epsilon)\xi(t)$$

$$(\pi(0,s,0)\xi)(t) = \xi(t+s).$$

Consequently

$$\pi(0,0,t)\xi = \bar{e}(t/\epsilon)\xi .$$

Notice that the definitions of ∇ and π do not involve w_1 and w_2, so that π is just dim(K)-copies of the usual Schrödinger representation of H on $L^2(\mathbb{R})$, once allowance is made for ϵ. Furthermore, Ξ consists of exactly the C^∞-vectors for π, as discussed in [H].

As before, let $E = \mathrm{End}_A(\Xi)$, with E_s its skew-adjoint part. As seen earlier, any other compatible connection on Ξ will be of the form $\nabla + \omega$ where ω is an E-valued one-form on L. Now A_θ has a canonical trace τ, which takes $\Sigma f(m,n)U_1^m U_2^n$ to $f(0,0)$. A glance at the definition of $< \, , \, >_A$ shows that the ordinary inner-product of $L^2(\mathbb{R},K)$ is just $\tau(< \, , \, >_A)$. Since the elements of E are bounded operators with respect to $< \, , \, >_A$, it follows that they are bounded for the L^2-norm. In particular, the values of ω are bounded operators for the L^2-norm, so that $\nabla + \omega$ is a perturbation of ∇ by bounded operators.

Suppose now that $\nabla + \omega$ minimizes YM. Then Theorem 2.1 tells us that $\nabla + \omega$ must be of constant curvature, and this curvature must be exactly the same as that for ∇. It follows that $\nabla + \omega$ defines a representation of the Heisenberg commutation relations of the same form as ∇. We wish to determine how $\nabla + \omega$ is related to ∇ by the gauge group, and for this we need to know that the representation determined by $\nabla + \omega$ exponentiates to give a unitary representation of H on $L^2(\mathbb{R},K)$. In preparation for this, we note that, as mentioned before, ∇ defines a covariant derivative, $\hat{\delta}$, on E, by

$$\hat{\delta}_X(T) = [\nabla_X,T]$$

for $T \in E$ and $X \in L$. For our immediate purposes all we need is that $\hat{\delta}_X(T)$ is in E for every $T \in E$. This is trivial to verify.

This puts us in position to apply Theorem 9.9c of [JM], with E playing the role of the P of that theorem. The key hypothesis of the theorem is that the perturbation of the Lie algebra be by bounded operators whose *commutants* with every element of the Lie algebra are again *bounded*. Since we are perturbing by ω, whose values are in E, and since commutants with elements of the Lie algebra generated by ∇_1 and ∇_2 are just what define $\hat{\delta}$, which carries E into itself, this key hypothesis is satisfied. Applying Theorem 9.9c of [JM], we conclude that there is a representation, ρ, of H on $L^2(\mathbb{R},K)$ whose space of C^∞-vectors is again Ξ and whose infinitesimal form is determined by $\nabla + \omega$.

By the Stone-von Neumann theorem on the uniqueness of the Heisenberg commutation relations, ρ will be equivalent to a certain number of copies of the (irreducible) Schrödinger representation. We need to know this multiplicity. Now it is not difficult to verify that the operator $\nabla_1 + i\nabla_2$, when viewed just on $L^2(\mathbb{R})$, has index 1. Consequently, on $L^2(\mathbb{R},K)$ its index will be dim(K). But $(\nabla_1+\omega_1) + i(\nabla_2+\omega_2)$ is a perturbation of $\nabla_1 + i\nabla_2$ by a bounded operator, and it can be shown that such a perturbation will not change the index. Since $(\nabla_1+\omega_1) + i(\nabla_2+\omega_2)$ stands in the same position to ρ as $\nabla_1 + i\nabla_2$ does to π, we conclude that ρ contains the irreducible representation with multiplicity $\dim(K)$. In particular, π and ρ are unitarily equivalent, and so there exists a unitary operator, Q, on $L^2(\mathbb{R},K)$ such that $\rho_x = Q\pi_x Q^*$ for all $x \in H$. Note that Q is not unique.

We need to show that Q carries Ξ into itself. Let π^j and ρ^j denote the restrictions of π and ρ to the one-parameter groups with generators ∇_j and $\nabla_j + \omega_j$ for $j = 1,2$. By the usual Phillips theory of perturbation of a single generator by a bounded operator, there is a C^∞ bounded-operator-valued function c^j (a cocycle) such that $\pi^j_t = c^j_t\rho^j_t$ for $t \in \mathbb{R}$. Then

$$\pi^j_t\, Q\pi^{j*}_t = \pi^j_t\, \rho^{j*}_t Q = c^j_t\, Q.$$

It follows that Q is a C^∞-vector for the action of H on the algebra of bounded operators on $L^2(\mathbb{R},K)$ obtained by conjugating by π. Let $\xi \in \Xi$. Since Ξ consists of exactly the C^∞-vectors for π, and since

$$\pi_x(Q\xi) = (\pi_x Q\pi^*_x)(\pi_x\xi),$$

it follows that $Q\xi$ is a C^∞-vector for π and so is in Ξ. That is, Q carries Ξ into itself as desired. Thus Q will intertwine at the infinitesimal level, and we have

$$Q\nabla_X Q^* = \nabla_X + \omega_X$$

for all $X \in L$, as operators on Ξ.

Now Q need not be in E. But let us calculate the extent to which it fails to commute with the generators of A. For ease of notation, define operators W_j on Ξ by $W_j\xi = \xi U_j$ for $j = 1,2$. Note then that for $\xi \in \Xi$

$$\begin{aligned}(W_j^*\nabla_k W_j)\xi &= (\nabla_k(\xi U_j))U_j^* = (\nabla_k(\xi)U_j + \xi\delta_k(U_j))U_j^* \\ &= \nabla_k(\xi) + 2\pi i\delta_{jk}\xi,\end{aligned}$$

where δ_{jk} is the Kronecker delta. Thus

$$W_j^* \nabla_k W_j = \nabla_k + 2\pi i \delta_{jk} I.$$

Then

$$\begin{aligned}(W_j Q W_j^*) \nabla_k (W_j Q W_j^*)^* &= W_j Q(\nabla_k + 2\pi i \delta_{jk} I) Q^* W_j^* \\ &= W_j(\nabla_k + \omega_k) W_j^* + 2\pi i \delta_{jk} I \\ &= W_j \nabla_k W_j^* + \omega_k + 2\pi i \delta_{jk} I\end{aligned}$$

(since ω_k commutes with U_j and so with W_j)

$$= \nabla_k - 2\pi i \delta_{jk} I + \omega_k + 2\pi i \delta_{jk} I = \nabla_k + \omega_k.$$

It follows that $W_j Q W_j^*$ intertwines π and ρ. Consequently $Q^*(W_j Q W_j^*)$ intertwines π with itself. But because the Schrodinger representation on $L^2(\mathbb{R})$ is irreducible, the only unitary intertwining operators for π with itself are of the form $I \otimes u$ where $u \in U(K)$, the group of unitary operators on K, and where I is here the identity operator on $S(\mathbb{R})$. It follows that for $j = 1,2$, there is a (unique) $u_j \in U(K)$ such that

$$W_j\, Q W_j^* = Q(I \otimes u_j).$$

Now because of the commutation relation between U_1 and U_2, and so between W_1 and W_2, it is clear that

$$W_1 W_2 Q W_2^* W_1^* = W_2 W_1 Q W_1^* W_2^* .$$

If we use the above definition of u_j, these two sides become, respectively,

$$Q(I \otimes u_1 w_1 u_2 w_1^*) \quad \text{and} \quad Q(I \otimes u_2 w_2 u_1 w_2^*).$$

It follows that

$$u_1 w_1 u_2 w_1^* = u_2 w_2 u_1 w_2^* .$$

From the fact that $w_2 w_1 = e(-p/q) w_1 w_2$, we find that

$$(u_1 w_1)(u_2 w_2) = e(p/q)(u_2 w_2)(u_1 w_1).$$

That is, $u_1 w_1$ and $u_2 w_2$ satisfy the same commutation relation as w_1 and w_2.

Now K decomposes into a direct sum of subspaces, say $K_1, \ldots, K_d$, of dimension q on which w_1 and w_2 give an irreducible representation of the Heisenberg commutation relations for the group $\mathbb{Z}_q$. Similarly K decomposes into a direct sum of subspaces, $K_1', \ldots, K_d'$ on which $u_1 w_1$ and $u_2 w_2$ act irreducibly. It need not be true that $(u_1 w_1)^q = I_k = (u_2 w_2)^q$. However clearly $(u_1 w_1)^q$ and $(u_2 w_2)^q$ commute with $u_1 w_1$ and $u_2 w_2$. Thus on each K_k' the restrictions of $(u_1 w_1)^q$ and $(u_2 w_2)^q$ act as scalar multiples of the identity operator. Taking any qth roots of the corresponding scalars, we find complex numbers β_{jk} of modulus one such that $(\bar{\beta}_{jk} u_j w_j)^q$ acts as the identity operator on K_k' for $j = 1,2$. Thus $\bar{\beta}_{1k} u_1 w_1$ and $\bar{\beta}_{2k} u_2 w_2$ acting on K_k' define an irreducible representation of the Heisenberg commutation relations for $\mathbb{Z}_q$. Since all such representations are equivalent, we can find a (not unique) unitary operator, v_1 on K which simultaneously intertwines the representations on K_k and K_k' for each k, that is,

$$v^*(\bar{\beta}_{jk} u_j w_j) v = w_j \quad \text{on} \quad K_k$$

for $j = 1,2$ and all k. Let β_j be the operator on K which on K_k consists of multiplication by β_{jk}. The above relation then becomes

$$v^*(u_j w_j) v = \beta_j w_j$$

for $j = 1,2$, on all of K.

Let $\bar{Q} = Q(I \otimes v)$. Because π and ∇ do not involve the w_j's, it is clear that $\bar{Q}$ intertwines π and ρ, and that at the infinitesimal level

$$\bar{Q} \nabla_x \bar{Q}^* = \nabla_X + \omega_X.$$

But $\bar{Q}$ has a simpler relation to the W_j's than does Q, for

$$\begin{aligned} W_j\bar{Q}W_j^* &= (W_jQW_j^*)(W_j(I\otimes v)W_j^*) \\ &= Q(I\otimes u_j)(V_j\otimes w_j)(I\otimes v)(V_j\otimes w_j)^* \\ &= Q(I\otimes u_jw_jvw_j^*) = Q(I\otimes v\beta_j) = \bar{Q}(I\otimes\beta_j) \end{aligned}$$

for all $j = 1,2$.

For each k we can find a real number σ_1^k such that if M_k denotes pointwise multiplication on $S(\mathbb{R})$ by $t \to e(t\sigma_1^k)$ then $V_1M_kV_1^* = \bar{\beta}_{1k}M_k$. We can also find a real number σ_2^k such that if T_k denotes translation by $\epsilon\sigma_2^k$ then $V_2T_kV_2^* = \bar{\beta}_{2k}T_k$. Define the unitary operator N_k on $S(\mathbb{R},K_k)$ by $N_k = M_kT_k \otimes I_{Kk}$. Then on $S(\mathbb{R},K_k)$ we have

$$W_jN_kW_j^* = \bar{\beta}_{jk}N_k \quad \text{for } j = 1,2.$$

Let N be defined on $\Xi = S(\mathbb{R},K)$ to be the direct sum of the N_k's. Then

$$W_jNW_j^* = (I\otimes\beta_j)^*N.$$

Let $U = \bar{Q}N$. Then

$$W_jUW_j^* = (W_j\bar{Q}W_j^*)(W_jNW_j^*) = \bar{Q}(I\otimes\beta_j)(I\otimes\beta_j^*)N = U.$$

Thus $U \in E$, so that U is a gauge transformation. However U need not intertwine π and ρ. But simple calculations show that at the infinitesimal level we have on $S(\mathbb{R},K_k)$

$$N^*\nabla_jN = (M_kT_k\otimes I_{K_k})^*\nabla_j(M_kT_k\otimes I_{K_k}) = \nabla_j + 2\pi i\sigma_j^k I.$$

Let σ_j be the element of E_s which on each $S(\mathbb{R},K_k)$ is multiplication by $2\pi i\sigma_j^k$. Then on $S(\mathbb{R},K)$ we have

$$N^*\nabla_jN = \nabla_j + \sigma_j\,.$$

Consequently,

$$U(\nabla_j+\sigma_j)U^* = (UN^*)\nabla_j(UN^*)^* = \bar{Q}\nabla_j\bar{Q}^* = \nabla_j + \omega_j\,.$$

Note that every connection of the form $\nabla + \sigma$ for σ as above has the same curvature as ∇, and so does minimize YM.

We can summarize what we have discovered so far as follows.

3.1. PROPOSITION. *The subset of* $MC(\Xi)$ *consisting of connections of the form* $\nabla + \sigma$, *where* σ *is diagonal for the decomposition of* K *into irreducible representations and the diagonal entries of* σ *are imaginary multiples of the identity operators, meets every orbit of the action of* UE *on* $MC(\Xi)$.

4. THE MODULI SPACES FOR HEISENBERG MODULES

The space of connections of form $\nabla + \sigma$ for σ of the special type described in Proposition 3.1 is clearly an affine space whose vector space of translations is isomorphic to $\mathbb{R}^{2d}$, where d is the number of irreducible subspaces into which K decomposes. We must now determine when two elements of this affine space will be in the same orbit under the action of the gauge group.

For this purpose we use the fact that Ξ is isomorphic to d copies of the module obtained by requiring that w_1 and w_2 act irreducibly on K. It will be convenient now to change notation and let Ξ denote this module for irreducible K, so that we are concerned with Ξ^d. We need to obtain quite precise information about $B = \mathrm{End}_A(\Xi)$, since $E = M_d(B)$. Such precise information was already obtained in Theorem 1.1 of [R4], using the general framework discussed in [R2], but in the slightly more complicated situation where there is multiplicity, and with different notation. Rather than try to explain how to adapt Theorem 1.1 of [R4] to the present situation, it is easier just to rederive what we need from [R2].

For the vector space K we will take $C(\mathbb{Z}_q)$, the vector space of complex-valued functions on $\mathbb{Z}_q$, so that $S(\mathbb{R}) \otimes K = S(\mathbb{R} \times \mathbb{Z}_q)$. The group G of [R2] is then $\mathbb{R} \times \mathbb{Z}_q$, and as the subgroups H and K of [R2] we take those generated by $(\epsilon,[p])$ and $(1/q,[1])$ respectively. So the transformation group algebra $C^*(H,G/K)$ acts on the right on the completion of $S(\mathbb{R} \times \mathbb{Z}_q)$. The space G/K is identified with

$T = \mathbb{R}/\mathbb{Z}$ by the mapping $(t,[k]) \to t - k/q$. Let U_1 and U_2 denote the operators corresponding to the generator of H and to the function e on T. Then by formula 2.1 of [R2] we see that

$$(\xi U_1)(t,[k]) = \xi(t - \epsilon, [k - p])$$

$$(\xi U_2)(t,[k]) = e(t - k/q)\xi(t,[k]).$$

If we let w_1 and w_2 denote the operators on $C(\mathbb{Z}_q)$ consisting of translation by p and multiplication by $\bar{e}(k/q)$, we see that U_1 and U_2 are exactly the operators we have been using earlier.

Then B, at the completed C*-algebra level, will according to [R2] be exactly C*(K,G/H), acting on the left. The space G/H is identified with T by the mapping $(t,[k]) \to (t/\epsilon - ak)/q$, where a is an integer such that $ap + bq = 1$ for some integer b, using the fact that p and q are relatively prime ($a = 0$ if $p = 0$). Let Z_1 and Z_2 be the operators corresponding to the generator of K and to the function e on T. Then by the formula in 1.9 of [R2] we see that

$$(Z_1\xi)(t,[k]) = \xi(t - 1/q, [k-1])$$

$$(Z_2\xi)(t,[k]) = e((t/\epsilon - ak)/q)\xi(t,k).$$

It is easily verified that

$$Z_2 Z_1 = e(\psi) Z_1 Z_2$$

where $\psi = (a\theta - b)/(q\theta - p)$. Thus $B \cong A_\psi$.

The connection ∇ on Ξ and the representation π of the Heisenberg group will be defined by exactly the same formulas as before, in particular ignoring the variable in $\mathbb{Z}_q$. We notice that $\pi(r,0,0)$ will commute with Z_2 and $\pi(0,s,0)$ will commute with Z_1, while

$$\pi(r,0,0)Z_1\pi(-r,0,0) = e(r/q\epsilon)Z_1 ,$$

$$\pi(0,s,0)Z_2\pi(0,-s,0) = e(s/q\epsilon)Z_2 .$$

Thus conjugation by π is essentially the dual group action of T^2 on A_ψ. (In particular, we have a special case of the situation studied in [CM,CU].) Since $S(\mathbb{R} \times \mathbb{Z}_q)$ consists of exactly the C^∞-vectors for π, the endomorphism algebra for $S(\mathbb{R} \times \mathbb{Z}_q)$, rather than its completion, will be exactly the C^∞-vectors for the dual group action. From now on we let B denote this algebra of C^∞-vectors. Then, as with A_θ, every element of B is of the form

$$\sum_{m,n} \beta_{mn} Z_1^m Z_2^n$$

where β is a complex-valued function vanishing rapidly at infinity on $\mathbb{Z}^2$. It follows that every element of E will be an $n \times n$ matrix of such series.

We note that the infinitesimal form of the above commutation relations between π and Z_1 and Z_2 is

$$[\nabla_j, Z_k] = \gamma \delta_{jk} Z_k$$

where $\gamma = 2\pi i/\epsilon q$. We now consider an E-valued form σ as in Proposition 3.1, but to simplify computation, we take its diagonal entries to be of the form $\gamma\sigma_j^k$ for $j = 1,2$ and $k = 1, ..., d$, where the σ_j^k are real numbers. We consider the connection on Ξ^d determined by $\nabla + \sigma$.

In the same way let μ be a diagonal E-valued form with entries $\gamma\mu_j^k$. We want to determine when there will be a unitary U $\in$ E such that

$$U(\nabla + \sigma)U^* = \nabla + \mu,$$

or equivalently, such that

$$\nabla U - U\nabla = U\sigma - \mu U.$$

Let the entries of U be denoted by $U_{k\ell} \in B$. Then the last equation becomes

$$[\nabla_j, U_{k\ell}] = U_{k\ell}\gamma\sigma_j^\ell - \gamma\mu_j^k U_{k\ell} .$$

Now let $U_{k\ell}$ be given by

$$U_{k\ell} = \Sigma\, \beta^{k\ell}_{mn}\, Z_1^m Z_2^n .$$

Substituting this into the above equation, we obtain for $j = 1$

$$\Sigma\, \beta^{k\ell}_{mn}\, \gamma m Z_1^m Z_2^n = \Sigma\, \beta^{k\ell}_{mn}\, \gamma(\sigma_1^\ell - \mu_1^k) Z_1^m Z_2^n ,$$

and similarly for $j = 2$. Equating coefficients of $Z_1^m Z_2^n$ and dividing by γ, we obtain

$$\beta^{k\ell}_{mn}\,(m - (\sigma_1^\ell - \mu_1^k)) = 0$$

$$\beta^{k\ell}_{mn}\,(n - (\sigma_2^\ell - \mu_2^k)) = 0.$$

Thus if $U^{k\ell} \neq 0$, then $\sigma_j^\ell - \mu_j^k$ must be an integer.

Since U is unitary, and since $E = M_d(A_\psi)$ has a finite faithful trace, we can find a permutation, ρ, of the integers from 1 to d such that $U^{k\rho(k)} \neq 0$ for all k. It follows that σ_j^k and $\mu_j^{\rho(k)}$ differ by an integer, for $j = 1,2$. Thus we see that if $\nabla + \sigma$ and $\nabla + \mu$ are in the same orbit under the action of the gauge group, then μ is obtained from σ by first adding integer multiples of γ to the diagonal entries of σ_1 and σ_2, and then by permuting all these diagonals entries, simultaneously for σ_1 and σ_2.

Let us see that all such transitions from σ to μ are obtained from a gauge transformation. Now conjugation by ordinary permutation matrices in M_d does not change ∇, but permutes the diagonal entries of σ_1 and σ_2. So every permutation of the diagonal entries (simultaneously for σ_1 and σ_2) comes from a gauge transformation. On the other hand, suppose that for some integer m we wish to add m to the kth diagonal entry of σ_1, without changing σ_2 or any of the other entries of σ_1. Let U be the matrix which differs from the identity matrix only by having Z_1^{-m} as the kth diagonal entry. Then it is clear that conjugation by U does not change σ_2 or ∇_2, whereas from the commutation relation given above for ∇_1 and Z_1 we see that $U\nabla_1 U^*$ will be the sum of ∇_1 with the matrix having $m\gamma$ in the kth diagonal entry and 0's elsewhere.

By using Z_2 we can in the same way add integer multiples of γ to the entries of σ_2. Thus for each k the possible values of σ_1^k and σ_2^k modulo integer multiples of γ, contribute a two-torus T^2 to the moduli space. But permutations from the two-torus for one value of k to that for another value of k are permitted. Let Σ_d denote the permutation group on d symbols. We have obtained the following description of the moduli space:

4.1. THEOREM. *Let* p *and* q *be integers which are either relatively prime with* $q > 0$, *or* $p = 0$ *and* $q = 1$, *and assume that* $p/q \neq \theta$. *Let* Ξ_{pq} *denote* $S(\mathbb{R}) \otimes K$ *as projective right* A_θ*-module in the way described earlier, where* $\dim(K) = q$ *and the action on* K *is irreducible. Let* d *be any positive integer. Then the moduli space* $MC(\Xi^d)/UE$ *for compatible connections on* Ξ^d *which minimize the Yang-Mills functional, is homeomorphic to*

$$(T^2)^d/\Sigma_d ,$$

where the action of Σ_d *is by permutation of components.*

5. MODULI SPACES FOR OTHER PROJECTIVE MODULES

If θ is irrational, then it follows from the main theorem of [R4] that every finitely generated projective A_θ-module which is not free is isomorphic to a Heisenberg module. But the free modules are not Heisenberg modules. If θ is rational, then calculation of the dimension and Chern character of the Heisenberg modules shows that there are sometimes even non-free modules which are not Heisenberg modules. In this section we show, by means of a little trick, how to use the information which we have obtained about Heisenberg modules to calculate the moduli spaces for these non-Heisenberg modules.

Our trick is based on the way compatible connections with constant curvature behave under tensor products. For any projective A-module Ξ it will be notationally convenient in this section to denote by $MC(\Xi)$ the space of compatible connections with constant curvature on Ξ (or by $MC(\Xi,\delta)$ if we want to make clear the relevant action of L on A). Recall that if $\nabla \in MC(\Xi)$ then its covariant derivative, $\hat{\delta}$, on $B = \mathrm{End}_A(\Xi)$ defined by $\hat{\delta}_X(T) = [\nabla_X,T]$, is an action of L on B.

5.1. LEMMA. *Let δ be an action of L on a unital pre-C*-algebra A, let Ω be a projective right A-module with Hermitian metric, and let $\nabla \in MC(\Omega)$. Let $B = \mathrm{End}_A(\Omega)$, and let $\hat{\delta}$ be the covariant derivative on B for ∇. Let Ξ be a projective right B-module, and let ∇' be a connection on Ξ for $\hat{\delta}$. Define ∇'' on $\Xi \otimes_B \Omega$ by*

$$\nabla'' = \nabla' \otimes I_\Omega + I_\Xi \otimes \Omega,$$

that is,

$$\nabla''_X(\xi \otimes \omega) = \nabla'_X(\xi) \otimes \omega + \xi \otimes \nabla_X(\omega).$$

Then ∇'' is a connection on $\Xi \otimes_B \Omega$ for δ. If Θ, Θ' and Θ'' denote the curvatures of ∇, ∇' and ∇'', then

$$\Theta''(X,Y) = \Theta'(X,Y) \otimes I_\Omega + I_\Xi \otimes \Theta(X,Y).$$

In particular, if ∇' has constant curvature then so does ∇''. If Ξ has a Hermitian metric, and if ∇' is compatible with it, then ∇'' is compatible with the associated Hermitian metric on $\Xi \otimes_B \Omega$.

Verification of the above statements is entirely straightforward, but we should clarify that by the "associated Hermitian metric" we mean that defined, as in Theorem 5.9 of [R1], by

$$\langle \xi \otimes \omega, \eta \otimes \zeta\rangle_A = \langle \omega, \langle \eta,\xi\rangle_B \zeta\rangle_A$$

for $\xi,\eta \in \Xi$ and $\omega,\zeta \in \Omega$.

We also need dual modules, as defined in 6.17 of [R1], and connections on them. For this it will be convenient to assume that the projective modules are *full*. For an A-module Ω with Hermitian metric this means that the span of the range of $\langle\ ,\ \rangle_A$ is all of A, so that Ω establishes a Morita equivalence between A and $B = \mathrm{End}_A(\Omega)$.

5.2. LEMMA. *Let* A, δ, Ω, ∇, B *and* $\hat{\delta}$ *be as in the previous lemma, but assume now that* Ω *is full. Let* $\tilde{\Omega}$ *denote the dual module for* Ω, *that is,* $\tilde{\Omega}$ *is the additive group* Ω, *but made into an* A-B-*bimodule by*

$$a\tilde{\omega} = (\omega a^*)^{\sim}, \quad \tilde{\omega}b = (b^*\omega)^{\sim}.$$

Define $\tilde{\nabla}$ *on* $\tilde{\Omega}$ *by* $\tilde{\nabla}_X(\tilde{\omega}) = (\nabla_X(\omega))^{\sim}$. *Then* $\tilde{\nabla} \in MC(\tilde{\Omega},\hat{\delta})$, *and the covariant derivative of* $\tilde{\nabla}$ *is* δ *on* A.

As explained in Lemma 6.2 of [R1], if Ω is full then $\tilde{\Omega} \otimes_B \Omega$ is naturally identified with A as A-A-bimodules by $\tilde{\zeta} \otimes \omega \to \langle\xi,\omega\rangle_A$, while $\Omega \otimes_A \tilde{\Omega}$ is naturally identified with B as B-B-bimodules by $\omega \otimes \tilde{\zeta} \to \langle\omega,\zeta\rangle_B$. We need the corresponding facts for connections, when in the next lemma δ is viewed as a connection on the free A-module A, and similarly for $\hat{\delta}$.

5.3. LEMMA. *Let* A, δ, Ω, ∇, B *and* $\hat{\delta}$ *be as above, with* Ω *full. Then*

$$\nabla \otimes I_{\tilde{\Omega}} + I_\Omega \otimes \tilde{\nabla}$$

is identified with $\hat{\delta}$ *under the natural identification of* $\Omega \otimes_A \tilde{\Omega}$ *with* B, *while*

$$\tilde{\nabla} \otimes I_\Omega + I_{\tilde{\Omega}} \otimes \nabla$$

is identified with δ *under the natural identification of* $\tilde{\Omega} \otimes_B \Omega$ *with* A.

PROOF. $(\nabla_X \otimes I_{\tilde{\Omega}} + I_\Omega \otimes \tilde{\nabla}_X)(\omega \otimes \tilde{\zeta}) = \nabla_X(\omega) \otimes \tilde{\zeta} + \omega \otimes (\nabla_X\zeta)^{\sim}$, which is identified with

$$\langle\nabla_X(\omega),\zeta\rangle_B + \langle\omega,\nabla_X\zeta\rangle_B = \tilde{\delta}_X(\langle\omega,\zeta\rangle_B).$$

The proof of the other identification is similar. Q.E.D.

5.4. PROPOSITION. *Let* A, δ, Ω, ∇, B *and* $\hat{\delta}$ *be as above, and let* Ξ *be a projective* B-*module with Hermitian metric. Consider the map which sends any* ∇' *in* $MC(\Xi,\hat{\delta})$ *to*

$$\nabla' \otimes I_\Omega + I_\Xi \otimes \nabla$$

in $MC(\Xi \otimes_B \Omega,\delta)$. *If* Ω *is full, then this map has a two-sided inverse* *from* $MC(\Xi \otimes_B \Omega, \delta)$ *to* $MC(\Xi, \hat{\delta})$.

PROOF. Suppose that Ω is full, so that $\tilde{\Omega}$ and $\tilde{\nabla}$ are defined. Our inverse will be the map which sends $\nabla'' \in MC(\Xi \otimes_B \Omega, \delta)$ to

$$\nabla'' \otimes I_{\tilde{\Omega}} + I_{\Xi\otimes\Omega} \otimes \tilde{\nabla}$$

in $MC(\Xi \otimes_B \Omega \otimes_A \tilde{\Omega}, \tilde{\delta})$, but with the latter identified with $MC(\Xi, \tilde{\delta})$ by means of the identification of $\Xi \otimes_B \Omega \otimes_A \tilde{\Omega}$ with Ξ_B. Under this map $\nabla' \otimes I_\Omega + I_\Xi \otimes \nabla$ is sent to

$$\nabla' \otimes I_\Omega \otimes I_{\tilde{\Omega}} + I_\Xi \otimes \nabla \otimes I_{\tilde{\Omega}} + I_\Xi \otimes I_\Omega \otimes \tilde{\nabla},$$

which by Lemma 5.3 is identified with

$$\nabla' \otimes I_B + I_\Xi \otimes \tilde{\delta}\,,$$

on $\Xi \otimes_B B$, which is easily seen to be identified with ∇' by the map $\xi \otimes b \to \xi b$. Thus we have a left inverse. On the other hand,

$$\nabla'' \otimes I_{\tilde{\Omega}} + I_{\Xi\otimes\Omega} \otimes \tilde{\nabla}$$

is sent to

$$\nabla'' \otimes I_{\tilde{\Omega}} \otimes I_\Omega + I_{\Xi\otimes\Omega} \otimes \tilde{\nabla} \otimes I_\Omega + I_{\Xi\otimes\Omega\otimes\tilde{\Omega}} \otimes \nabla\,,$$

which by Lemma 5.3 is identified with

$$\nabla'' \otimes I_A + I_{\Xi\otimes\Omega} \otimes \delta,$$

which is easily seen to be identified with ∇'' by the map $(\xi \otimes \omega) \otimes a \to \xi \otimes (\omega a)$. Thus we have a right inverse. Q.E.D.

5.5. THEOREM. *Let* A, δ, Ω, ∇, B *and* $\hat{\delta}$ *be as above. Let* Ξ *be a projective* B*-module with Hermitian metric, let* $E = End_B(\Xi)$, *and assume that* Ω *is full so that also* $E = End_A(\Xi \otimes_B \Omega)$. *Thus the gauge group* UE *acts on both* $MC(\Xi, \hat{\delta})$ *and* $MC(\Xi \otimes_B A, \delta)$. *Then the bijection used above which sends* $\nabla' \in MC(\Xi, \tilde{\delta})$ *to*

$$\nabla' \otimes I_\Omega + I_\Xi \otimes \nabla$$

in $MC(\Xi \otimes_B \Omega,\delta)$ *is equivariant for the action of* UE, *and so gives a bijection between the corresponding orbit spaces, that is, between the corresponding moduli spaces.*

PROOF. Let $u \in UE$ and let $\nabla' \in MC(\Xi)$. Then $(\gamma_u(\nabla'))_X = u\nabla'_X u^*$, which is sent to

$$u\nabla'_X u^* \otimes I_\Omega + I_\Xi \otimes \nabla_X$$
$$= (u \otimes I_\Omega)(\nabla'_X \otimes I_\Omega + I_\Xi \otimes \nabla_X)(u \otimes I_\Omega)^*. \qquad \text{Q.E.D.}$$

In order to exploit this theorem, we must arrange matters so that Ξ is a Heisenberg module, for then we can apply the results of the previous sections. Before embarking on this, we remark that any (non-zero) projective module over any A_θ is full. If θ is irrational this follows from the fact that A_θ is simple, while if θ is rational this follows from the fact that A_θ is then Morita equivalent to $C(T^2)$.

5.6. PROPOSITION. *Let* θ *be any real number, let* $A = A_\theta$, *and let* Λ *be a projective* A-*module. Then we can factor* Λ *as*

$$\Lambda \cong \Xi^d \otimes_B \Omega$$

for some positive integer d, *where* Ω, B, *and* Ξ *have the following properties. First,* Ω *is a projective* A-*module and* $B = \mathrm{End}_A(\Omega)$. *Also there is a real number* φ *such that* B *is identified with* A_φ. *Next, there is a Hermitian metric on* Ω *and a* $\nabla \in MC(\Omega)$ *such that the convariant derivative* $\hat{\delta}$ *on* B *defined by* ∇ *is, up to a scale factor, the usual action of* L *on* A_φ. *Furthermore,* Ξ *is a Heisenberg* B-*module, and* $\Xi \otimes_B \Omega$ *is not a multiple of any other projective* A-*module. In particular, every projective* A_θ-*module admits a compatible connection with constant curvature.*

PROOF. By considering the positive cone of $K_0(A_\theta)$, which for θ rational looks like that for $K_0(C(T^2))$ since A_θ is Morita equivalent to $C(T^2)$, and which for θ irrational is shown in [PV] to be $(\mathbb{Z} + \mathbb{Z}\theta) \cap [0,\infty)$, we see that every projective A-module is a multiple of some projective module which itself is not a multiple of any other projective module. Thus it suffices to prove the proposition for the case in which Λ is not a multiple of any other projective A-module (so $d = 1$).

Suppose first that θ is irrational. If Λ is not free then, as indicated earlier, Λ is a Heisenberg A-module, and so we can let $\Xi = \Lambda$, $B = A$ and $\Omega = A$, with connection δ on Ω.

Suppose next that θ is rational or irrational, but that Λ is free. Then by the multiplicity assumption Λ must be A itself. From Theorem 1.1 of [R4] it follows that we can find a real number φ such that, with $B = A_\varphi$, there is a Heisenberg B-module Ξ such that $A = \mathrm{End}_B(\Xi)$. Let ∇ be the standard connection on Ξ, so that, as seen in Section 4, the covariant derivative of ∇ on A induces the usual action of L on A up to a scale factor ϵq. By rescaling, we find that $\breve{\nabla}$ on $\breve{\Xi}$ is a compatible connection of constant curvature for the usual action of L on A, whose covariant derivative on B is the usual action of L on B up to a scalar factor. Thus it suffices to set $\Omega = \breve{\Xi}$.

Suppose now that θ is rational and that Λ is not free. Let $C = C(T^2)$. Again, considerations of dimension and Chern character together with the fact that stable isomorphism implies isomorphism, or examination of Theorem 3.1 and the discussion before Theorem 3.9 of [R4], shows that every projective C-module is either free or Heisenberg. Thus if $A = C$ we can argue exactly as for the case of θ irrational. If $A \breve{\neq} C$, then by Theorem 3.9 of [R4], there is a Heisenberg C-module Ω' such that $A \cong \mathrm{End}_C(\Omega')$. Let $\Omega = (\Omega')\breve{}$, with connection ∇ scaled so as to be a connection for δ, and with covariant derivative giving the usual action of L on C up to a scale factor. Since Ω is automatically full, Ω establishes a Morita equivalence of A with C, and consequently any projective A-module will be of the form $\Xi \otimes_C \Omega$ for some projective C-module Ξ. Choose Ξ so that $\Lambda \cong \Xi \otimes_C \Omega$. If Ξ is not free, then as stated above, Ξ is a Heisenberg C-module and we are done. If Ξ is free, then by the multiplicity assumption $\breve{\Xi} = C$, so $\Lambda \cong \Omega$. Then, as in the free case discussed above, we can find a $B = A_\varphi$ and a Heisenberg B-module, say Ξ, such that $C \cong \mathrm{End}_B(\Xi)$. Then

$$\Lambda \cong \Xi \otimes_B (\tilde{\Xi} \otimes_C \Omega).$$

A straightforward argument shows that the connection on $\tilde{\Xi} \otimes_C \Lambda$ coming from the usual connections on the Heisenberg modules Ω and Ξ will have the desired properties. Thus the proof is complete if we let $\tilde{\Xi} \otimes_C \Omega$ be our new Ω.

5.7. THEOREM. *Let θ be any real number, and let* d *be any positive integer. With $A = A_\theta$, consider any projective A-module of the form Λ^d where Λ is not a multiple of any other projective A-module. Let Λ^d be equipped with a Hermitian metric. Then the moduli space for the compatible connections which minimize* YM *is homeomorphic to* $(T^2)^d/\Sigma_d$.

PROOF. We apply Proposition 5.6 to write Λ as $\Xi \otimes_B \Omega$ where Ξ and Ω have the properties stated in the proposition. Then $\Lambda^d \cong \Xi^d \otimes_B \Omega$. We saw in Section 1 that the moduli space for YM does not depend on the choice of Hermitian metric, and so we can change that on Λ to be the one coming from the Hermitian metrics on Ξ and Ω. Let δ' be the usual action of L on A_φ, so that $\hat{\delta} = \alpha\delta'$ for some positive number α. Then a moment's thought shows that multiplication by α gives a UE-equivariant map of $MC(\Xi^d,\delta')$ to $MC(\Xi^d,\hat{\delta})$, where curvatures are correspondingly scaled by α^2, and so YM is scaled by α^4. Thus the moduli space for $MC(\Xi^d,\hat{\delta})$ is naturally identified with that for $MC(\Xi^d,\delta')$. But by Theorem 6.6 the moduli space for $MC(\Lambda^d,\delta)$ is identified with that for $MC(\Xi^d,\hat{\delta})$, while by Theorem 4.1 the moduli space for $MC(\Xi^d,\delta')$ is homeomorphic to $(T^2)^d/\Sigma_d$. Q.E.D.

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