CHERN CHARACTER FOR DISCRETE GROUPS

by

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Dedicated to Professor Itiro Tamura on his sixtieth birthday

Let Γ be a group acting by homeomorphisms on a topological space X. View Γ as topologized by the discrete topology in which each point of Γ is an open set. Take the given action of Γ on X to be a right action so that the map

$$X \times \Gamma \longrightarrow X$$

which takes (x,γ) to $x\gamma$ is continuous.

Problem: How should we define the equivariant K theory $K^*_\Gamma(X)$ and the equivariant cohomology $H^*(X,\Gamma)$?

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If X is compact and Hausdorff there is the usual (i.e. non-equivariant) Chern character

$$ch : K^0(X) \longrightarrow \prod_{j\in\mathbb{N}} H^{2j}(X,\mathbb{Q})$$

$$ch : K^1(X) \longrightarrow \prod_{j\in\mathbb{N}} H^{2j+1}(X;\mathbb{Q})$$

$H^j(X;\mathbb{Q})$ is the j-th Čech cohomology group of X with coefficients the rational numbers $\mathbb{Q}$. The key property of this classical Chern character is that it is a rational isomorphism. Thus for any compact Hausdorff space X

$$ch : K^0(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \prod_{j\in\mathbb{N}} H^{2j}(X;\mathbb{Q})$$

$$ch : K^1(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \prod_{j\in\mathbb{N}} H^{2j+1}(X;\mathbb{Q})$$

are isomorphisms of vector spaces over $\mathbb{Q}$.

In the equivariant case $K^i_\Gamma(X)$ will be an abelian group, but $H^i(X,\Gamma)$ will be a vector space over the complex numbers $\mathbb{C}$. We require that the equivariant Chern character

$$ch_\Gamma : K^i_\Gamma(X) \longrightarrow H^i(X,\Gamma)$$

give an isomorphism of vector spaces over $\mathbb{C}$:

$$K^i_\Gamma(X) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow H^i(X,\Gamma) \qquad i = 0,1.$$

At this point there are difficulties with the traditional homotopy quotient approach to equivariant cohomology. Roughly speaking, the homotopy quotient theory is localized at the identity element of the group Γ. If X is a C^{∞} manifold and the action of Γ on X is C^{∞} and proper, then cyclic cohomology [16] can be used to define the "delocalized" equivariant cohomology of X.

In this note we give a direct geometrically-defined cohomology (denoted $H^{i}(X,\Gamma)$, $i = 0,1$) which for proper C^{∞} actions of Γ is isomorphic to the cyclic cohomology of the relevant algebra. We then exploit this isomorphism to solve the index problem for a Γ-equivariant family of elliptic operators parametrized by a C^{∞} manifold X on which the action of Γ is C^{∞} and proper.

Motivated by the index formula so obtained, we drop the hypothesis that the action of Γ on X is proper. Assuming only that X is a C^{∞} manifold on which Γ acts by diffeomorphisms we construct a commutative diagram:

$$\begin{array}{ccc} K^{i}(X,\Gamma) & \longrightarrow & K_{i}[C_{0}(X) \times \Gamma] \\ \downarrow & & \downarrow \\ H^{i}(X,\Gamma) & \longrightarrow & K_{i}[C_{0}(X) \times \Gamma] \otimes_{\mathbb{Z}} \mathbb{C} \end{array}$$

Here $C_{0}(X)$ is the C^{*} algebra of all continuous complex-valued functions on X which vanish at infinity. $C_{0}(X) \times \Gamma$ is the reduced crossed-product C^{*} algebra arising from the action of Γ on $C_{0}(X)$. We conjecture that both horizontal maps in this diagram are isomorphisms.

Of special interest is the case when X is a point. $C_{0}(\cdot) \times \Gamma$ is the reduced C^{*} algebra $C_{r}^{*}\Gamma$. Let $S(\Gamma)$ be the set of all elements in Γ of finite order. $F\Gamma$ is the permutation module (with coefficients $\mathbb{C}$) determined by the conjugation action of Γ on $S(\Gamma)$. As usual $H_j(\Gamma, F\Gamma)$ is the j-th homology of Γ with coefficients in $F\Gamma$.

$$H^0(\cdot, \Gamma) = \bigoplus_{j \in \mathbb{N}} H_{2j}(\Gamma, F\Gamma)$$

$$H^1(\cdot, \Gamma) = \bigoplus_{j \in \mathbb{N}} H_{2j+1}(\Gamma, F\Gamma)$$

So for X a point the above conjectured isomorphism becomes:

$$\bigoplus_{j \in \mathbb{N}} H_{2j}(\Gamma, F\Gamma) \cong K_0[C_r^*\Gamma] \otimes_{\mathbb{Z}} \mathbb{C}$$

$$\bigoplus_{j \in \mathbb{N}} H_{2j+1}(\Gamma, F\Gamma) \cong K_1[C_r^*\Gamma] \otimes_{\mathbb{Z}} \mathbb{C}$$

If true, this implies that the Novikov conjecture [15,29] and the Gromov-Lawson-Rosenberg conjecture [19,20] are true.

When Γ is torsion free $K^i(\cdot, \Gamma) = K_i(B\Gamma)$, the K homology [9] of the classifying space $B\Gamma$. In this case the conjectured isomorphism becomes

$$K_i(B\Gamma) \cong K_i[C_r^*\Gamma] \qquad i = 0,1$$

If true, this implies that for Γ torsion free there are no non-trivial idempotents in $C_r^*\Gamma$. This assertion (which is known as the generalized Kadison conjecture) is a much stronger statement than the classical conjecture [27] that for Γ torsion free there are no non-trivial idempotents in the group algebra $\mathbb{C}(\Gamma)$.

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This note is expository. We shall carefully outline the proofs of some, but not all, of the stated results. Complete proofs and details will appear elsewhere. As indicated above our aim is to show how index theory leads to the conjectured isomorphism $K^i(X,\Gamma) \cong K_i[C_0(X)\rtimes\Gamma]$.

§1. Γ finite: Chern character

Let Γ be a finite group acting by a continuous (right) action on a locally compact, Hausdorff, and paracompact topological space X.

$$X \times \Gamma \longrightarrow X \tag{1.1}$$

$H_c^*(X;\mathbb{C})$ is the Čech cohomology of X with compact supports.

$$H_c^*(X;\mathbb{C}) = \prod_{j\in\mathbb{N}} H_c^j(X;\mathbb{C}).$$

X^+ denotes the one-point compactification of X. For $j > 0$ the inclusion map $H_c^j(X;\mathbb{C}) \longrightarrow H^j(X^+;\mathbb{C})$ is an isomorphism. For $j = 0$ there is the exact sequence

$$0 \longrightarrow H_c^0(X;\mathbb{C}) \longrightarrow H^0(X^+;\mathbb{C}) \longrightarrow \mathbb{C} \longrightarrow 0 \tag{1.2}$$

$H_c^*(X;\mathbb{C})^\Gamma$ denotes those elements of $H_c^*(X;\mathbb{C})$ which are fixed by the action of Γ on $H_c^*(X;\mathbb{C})$.

Define $\hat{X} \subset X \times \Gamma$ by:

$$\hat{X} = \{(x,\gamma) \in X \times \Gamma \mid x\gamma = x\} \tag{1.3}$$

Γ acts on $\hat{X}$:

$$(x,\gamma)\alpha = (x\alpha,\alpha^{-1}\gamma\alpha) \qquad (x,\gamma) \in X \times \Gamma \tag{1.4}$$
$$\alpha \in \Gamma$$

Set

$$\begin{aligned} H^0(X,\Gamma) &= \prod_{j\in\mathbb{N}} H_c^{2j}(\hat{X}/\Gamma;\mathbb{C}) \\ H^1(X,\Gamma) &= \prod_{j\in\mathbb{N}} H_c^{2j+1}(\hat{X}/\Gamma;\mathbb{C}) \end{aligned} \tag{1.5}$$

For $\gamma \in \Gamma$, let $X^\gamma = \{x \in X \mid x\gamma = x\}$. $\hat{X}$ is the disjoint union of the X^γ.

$$\hat{X} = \bigcup_{\gamma\in\Gamma} X^\gamma$$

Hence

$$H_c^*(\hat{X}/\Gamma;\mathbb{C}) = H_c^*(\hat{X};\mathbb{C})^\Gamma = \Big[\bigoplus_{\gamma\in\Gamma} H_c^*(X^\gamma;\mathbb{C})\Big]^\Gamma \tag{1.6}$$

Equivariant K theory $K_\Gamma^i(X)$ $(i = 0,1)$ has been defined by M. F. Atiyah and G. B. Segal [4,30]. If X is compact, then $K_\Gamma^0(X)$ is the

Grothendieck group of Γ-vector-bundles on X. If X is not compact, extend the action of X to X^+, the one-point compactification of X, by requiring that the point at infinity be fixed by all $\gamma \in \Gamma$. Restriction to the point at infinity gives a map

$$K_\Gamma^0(X^+) \longrightarrow K_\Gamma^0(\cdot) = R(\Gamma) \tag{1.7}$$

where $R(\Gamma)$ is the representation ring of Γ. $K_\Gamma^0(X)$ is the kernel of this map. So by definition there is a short exact sequence

$$0 \longrightarrow K_\Gamma^0(X) \longrightarrow K_\Gamma^0(X^+) \longrightarrow R(\Gamma) \longrightarrow 0 \tag{1.8}$$

and an isomorphism

$$K_\Gamma^1(X) = K_\Gamma^1(X^+) \tag{1.9}$$

For X compact define the equivariant Chern character

$$ch_\Gamma : K_\Gamma^0(X) \longrightarrow H^0(X,\Gamma) \tag{1.10}$$

as follows. Suppose that E is a Γ-vector-bundle on X. If $x \in X^\gamma$, then E_x is mapped to itself by γ.

$$\gamma : E_x \longrightarrow E_x \qquad x \in X^\gamma \tag{1.11}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the (distinct) eigen-values of this linear transformation. If $i \neq j$, $\lambda_i \neq \lambda_j$ so that $E_x = E_x^1 \oplus E_x^2 \oplus \ldots \oplus E_x^r$ where E_x^i is the eigen-space for λ_i. At the vector-bundle level this gives a direct-sum decomposition

$$E \mid X^\gamma = E^1 \oplus E^2 \oplus \ldots \oplus E^r \tag{1.12}$$

where the action of γ on E^i is multiplication by λ_i. Define $ch_\Gamma^\gamma(E) \in \prod_{j\in\mathbb{N}} H^{2j}(X^\gamma;\mathbb{C})$ by:

$$ch_\Gamma^\gamma(E) = \sum_{i=1}^{r} \lambda_i ch(E^i) \tag{1.13}$$

In (1.13) $ch(E^i)$ is the ordinary (i.e. non-equivariant) Chern character of E^i. Then $ch_\Gamma : K_\Gamma^0(X) \longrightarrow \left[\bigoplus_{\gamma\in\Gamma} H_c^*(X^\gamma;\mathbb{C})\right]^\Gamma$ is:

$$ch_\Gamma(E) = \bigoplus_{\gamma\in\Gamma} ch_\Gamma^\gamma(E) \tag{1.14}$$

Compare [33].

More generally, X may be non-compact. To define ch_Γ : $K_\Gamma^i(X) \longrightarrow H^i(X,\Gamma)$, let Γ_γ denote the subgroup of Γ generated by γ. $tr_\gamma : R(\Gamma_\gamma) \longrightarrow \mathbb{C}$ is:

$$tr_\gamma(\varphi) = tr(\varphi(\gamma)). \quad \varphi \in R(\Gamma_\gamma) \tag{1.15}$$

Thus if $\varphi : \Gamma_\gamma \longrightarrow GL(n,\mathbb{C})$ is a representation of Γ_γ, then $tr_\gamma(\varphi)$ is the usual trace of the matrix $\varphi(\gamma)$.

The action of Γ_γ on X^γ is trivial so there is the isomorphism $K_{\Gamma_\gamma}^i(X^\gamma) = K^i(X^\gamma) \otimes_{\mathbb{Z}} R(\Gamma_\gamma)$. Using this isomorphism, restriction from (X,Γ) to (X^γ,Γ_γ) gives a map

$$\Pi^\gamma : K_\Gamma^i(X) \longrightarrow K^i(X^\gamma) \otimes_{\mathbb{Z}} R(\Gamma_\gamma) \tag{1.16}$$

Let $ch^{\gamma} : K^{i}(X^{\gamma}) \underset{\mathbb{Z}}{\otimes} R(\Gamma_{\gamma}) \longrightarrow H_{c}^{*}(X^{\gamma};\mathbb{C})$ be

$$ch^{\gamma} = ch \otimes tr_{\gamma} \tag{1.17}$$

In (1.17) $ch : K^{i}(X^{\gamma}) \longrightarrow H_{c}^{*}(X^{\gamma};\mathbb{C})$ is the ordinary (i.e. non-equivariant) Chern character. For $\xi \in K_{\Gamma}^{i}(X)$ define $ch_{\Gamma}(\xi) \in \left[\bigoplus_{\gamma\in\Gamma} H_{c}^{*}(X^{\gamma};\mathbb{C})\right]^{\Gamma}$ by

$$ch_{\Gamma}(\xi) = \bigoplus_{\gamma\in\Gamma} ch^{\gamma}\Pi^{\gamma}(\xi) \tag{1.18}$$

(1.19) *Theorem.* Let Γ be a finite group acting by homeomorphisms on a locally compact, Hausdorff, and paracompact space X. Then for $i = 0,1$ $ch_{\Gamma} : K_{\Gamma}^{i}(X) \longrightarrow H^{i}(X,\Gamma)$ gives an isomorphism of vector spaces over $\mathbb{C}$ $K_{\Gamma}^{i}(X) \underset{\mathbb{Z}}{\otimes} \mathbb{C} \longrightarrow H^{i}(X,\Gamma)$.

Remarks. $H^{i}(X,\Gamma)$ is defined by (1.5). In constructing ch_{Γ} the isomorphism (1.6) has been used. The proof of (1.19) will be outlined in the next section.

§2. Γ finite: Sheaf theory

X, Γ are as in §1.

The proof of Theorem (1.19) uses the Segal spectral sequence [30, 31]. The underlying idea of the proof is that both $K_{\Gamma}^{*}(X) \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ and $H^{*}(X,\Gamma)$ have a Mayer-Vietoris exact sequence. The two agree locally, and therefore are isomorphic. To give a precise argument we shall need a sheaf-theoretic interpretation of $H^*(X,\Gamma)$.

For $x \in X$, I_x denotes the isotropy group at x.

$$I_x = \{\gamma \in \Gamma \mid x\gamma = x\} \tag{2.1}$$

(2.2) *Lemma*. Given any $x \in X$, there exists an open subset U of X with $x \in U$ and with $I_p \subset I_x$ for all $p \in U$.

$$I_p \subset I_x$$

Proof. Let $\mathcal{U}$ be the collection of all open subsets U of X with $x \in U$. Suppose the lemma is false. Then for each $U \in \mathcal{U}$ there exists $(p_U, \gamma_U) \in U \times \Gamma$ with $p_U\gamma_U = p_U$ and $\gamma_U \notin I$. Since Γ is finite we may choose $\mathcal{V} \subset \mathcal{U}$ with $\mathcal{V}$ cofinal in $\mathcal{U}$ and with γ_U constant for $U \in \mathcal{V}$. Set $\underline{\gamma} = \gamma_U$ for $U \in \mathcal{V}$. Then $\varinjlim_{U=\mathcal{V}} p_U = p$ so $p\underline{\gamma} = p$. Since $\underline{\gamma} \notin I_x$ this is a contradiction. Q.E.D.

Using lemma (2.2) we can now define a sheaf $R(I)$ on X. For $x \in X$ the stalk of $R(I)$ is $R(I_x)$, the representation ring of I_x. If W is an open subset of X, then a (continuous) section of $R(I)$ on W is a function s which assigns to each $x \in W$, $s(x) \in R(I_x)$ satisfying:

(2.3) For each $x \in W$ there exists an open set U with $x \in U$ and with $I_p \subset I_x$ for all $p \in U$ and with $s(p) = \rho s(x)$ for all $p \in U$ where $\rho : R(I_x) \longrightarrow R(I_p)$ is the restriction map.

$R(I)$ is a Γ-sheaf [21]. Let $\varphi : I_x \longrightarrow GL(n,\mathbb{C})$ be a representation of $R(I_x)$. For $\gamma \in \Gamma$ let $\varphi\gamma \in R(I_{x\gamma})$ be

$$(\varphi\gamma)(\alpha) = \varphi(\gamma\alpha\gamma^{-1}) \qquad \alpha \in I_{x\gamma} \tag{2.4}$$

Then $\varphi \longmapsto \varphi\gamma$ is the (right) action of Γ on $R(I)$. Since I_x acts trivially on $R(I_x)$, the Γ-sheaf $R(I)$ descends to give a sheaf $\underline{R}(I)$ on X/Γ.

Set $R_{\mathbb{C}}(I_x) = R(I_x) \otimes_{\mathbb{Z}} \mathbb{C}$. By the same construction there is a Γ-sheaf $R_{\mathbb{C}}(I)$ on X whose stalk at $x \in X$ is $R_{\mathbb{C}}(I)$. $R_{\mathbb{C}}(I)$ descends to give a sheaf $\underline{R}_{\mathbb{C}}(I)$ on X/Γ. Denote the i-th cohomology group with compact supports of X/Γ using the sheaf $\underline{R}_{\mathbb{C}}(I)$ by $H^i_c(X/\Gamma;\underline{R}_{\mathbb{C}}(I))$.

(2.5) *Lemma*. For each $i = 0,1,2,\ldots$ there is a canonical isomorphism $H^i_c(\hat{X}/\Gamma;\mathbb{C}) \cong H^i_c(X/\Gamma;\underline{R}_{\mathbb{C}}(I))$.

Proof. According to (1.3) $\hat{X} \subset X \times \Gamma$. For $(x,\gamma) \in \hat{X}$, set $\Pi_1(x,\gamma) = x$. $\Pi : \hat{X}/\Gamma \longrightarrow X/\Gamma$ is the map of quotient spaces determined by Π_1. Consider the Leray spectral sequence [21,24] (with compact supports) of Π. This spectral sequence converges to $H^*_c(\hat{X}/\Gamma;\mathbb{C})$. The E_2 term is:

$$E_2^{p,q} = H^p_c(X/\Gamma;\underline{H}^q(\Pi^{-1}y)) \tag{2.6}$$

In (2.6) $\underline{H}^q(\Pi^{-1}y)$ denotes the sheaf on X/Γ whose stalk at $y \in X/\Gamma$ is $H^q(\Pi^{-1}y;\mathbb{C})$. Each $\Pi^{-1}y$ is a finite set, so:

$$\underline{H}^q(\Pi^{-1}y) = 0 \quad \text{for} \quad q > 0 \tag{2.7}$$

$c(I_x)$ denotes the set of conjugacy classes of I_x. There is the standard identification $R(I_x) \otimes_{\mathbb{Z}} \mathbb{C} = H^0(c(I_x);\mathbb{C})$. This identification gives an isomorphism of sheaves on X/Γ:

$$\underline{H}^0(\Pi^{-1}y;\mathbb{C}) = \underline{R}_{\mathbb{C}}(I) \tag{2.8}$$

(2.6), (2.7), and (2.8) prove the lemma. Q.E.D.

Proof of (1.19). G. B. Segal [30,31] has constructed a spectral sequence which converges to $K^*_\Gamma(X)$ and has for its E_2 term $H^*_c(X/\Gamma;\underline{R}(I))$. Similarly there is a spectral sequence converging to $K^*_\Gamma(X) \otimes_{\mathbb{Z}} \mathbb{C}$ with E_2 term $H^*_c(X/\Gamma;\underline{R}_{\mathbb{C}}(I))$. Also, there is a Segal spectral sequence for $H^*(X,\Gamma)$, but this spectral sequence is trivial and has $E_2 = E_\infty = H^*(X,\Gamma)$.

Consider the map of Segal spectral sequences induced by ch_Γ : $K^*(X,\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow H^*(X,\Gamma)$. According to the preceding lemma this is an isomorphism at the E_2 level. Since the Segal spectral sequence for $H^*(X,\Gamma)$ is trivial, this implies that $E_2 = E_\infty$ in the Segal spectral sequence for $K^*_\Gamma(X) \otimes_{\mathbb{Z}} \mathbb{C}$. Therefore at the level of $E_2 = E_\infty$, ch_Γ : $K^*_\Gamma(X) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow H^*(X,\Gamma)$ is an isomorphism, and *a fortiori* ch_Γ : $K^*_\Gamma(X) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow H^*(X,\Gamma)$ is itself an isomorphism. Q.E.D.

Remark. Set $H^0_{\mathbb{Z}}(X,\Gamma) = \prod_{j\in\mathbb{N}} H^{2j}_c(X/\Gamma;\underline{R}(I))$ $H^1_{\mathbb{Z}}(X,\Gamma) = \prod_{j\in\mathbb{N}} H^{2j+1}_c(X/\Gamma;\underline{R}(I))$. By lemma (2.5) $H^i_c(\hat{X}/\Gamma;\mathbb{C}) = H^i_c(X/\Gamma;\underline{R}_{\mathbb{C}}(I))$. Therefore the evident sheaf map $\underline{R}(I) \longrightarrow \underline{R}_{\mathbb{C}}(I)$ induces a map $H^i_{\mathbb{Z}}(X,\Gamma) \longrightarrow H^i(X,\Gamma)$. Thus we have:

$$\begin{tikzcd} & H^i_{\mathbb{Z}}(X,\Gamma) \arrow[d] \\ K^i_\Gamma(X) \arrow[r, "ch_\Gamma"'] & H^i(X,\Gamma) \end{tikzcd}$$

This is the analog of the standard maps in the non-equivariant case.

$$\begin{tikzcd} & H^*_c(X,\mathbb{Z}) \arrow[d] \\ K^*(X) \arrow[r, "ch"'] & H^*_c(X,\mathbb{Q}) \end{tikzcd}$$

§3. Γ finite: Homotopy quotient

X, Γ are as in §1.

The traditional approach to equivariant cohomology is to use $X \underset{\Gamma}{\times} E\Gamma$, the homotopy quotient. $E\Gamma$ is a contractible space on which Γ acts freely

$$E\Gamma \times \Gamma \longrightarrow E\Gamma \tag{3.1}$$

Γ acts on $X \times E\Gamma$ by the diagonal action.

$$(x,p)\gamma = (x\gamma, p\gamma) \qquad \begin{array}{l} x \in X \\ p \in E\Gamma \\ \gamma \in \Gamma \end{array} \tag{3.2}$$

$X \times_{\Gamma} E\Gamma$ is the quotient space.

$$X \times_{\Gamma} E\Gamma = (X \times E\Gamma)/\Gamma \tag{3.3}$$

If F is a Γ-vector-bundle on X, then $F \times_{\Gamma} E\Gamma = (F \times E\Gamma)/\Gamma$ is a vector bundle on $X \times_{\Gamma} E\Gamma$. $K^i(X \times_{\Gamma} E\Gamma)$ is the representable K theory.

$$\begin{aligned} K^0(X \times_{\Gamma} E\Gamma) &= [X \times_{\Gamma} E\Gamma, \mathbb{Z} \times BU] \\ K^1(X \times_{\Gamma} E\Gamma) &= [X \times_{\Gamma} E\Gamma, U] \end{aligned} \tag{3.4}$$

In (3.4) $[X,Y]$ is the set of homotopy classes of maps from X to Y, and $U = \lim_{n\to\infty} U(n)$.

For simplicity assume that X is compact. Map $K_{\Gamma}^0(X)$ to $H^*(X \times_{\Gamma} E\Gamma; \mathbb{C})$ by

$$F \longmapsto ch(F \times_{\Gamma} E\Gamma) \tag{3.5}$$

In (3.5) $ch : K^0(X \times_{\Gamma} E\Gamma) \longrightarrow H^*(X \times_{\Gamma} E\Gamma; \mathbb{C})$ is the ordinary (nonequivariant) Chern character. The map (3.5) is the traditional homotopy quotient Chern character.

We have the standard identification

$$H^*(X \times_{\Gamma} E\Gamma; \mathbb{C}) = H^*(X/\Gamma; \mathbb{C}) = H^*(X; \mathbb{C})^{\Gamma} \tag{3.6}$$

Granted (3.6), the traditional Chern character $F \longmapsto ch(F \times_{\Gamma} E\Gamma)$ is the composition

$$K_\Gamma^0(X) \xrightarrow{\mathrm{ch}_\Gamma} H^0(X,\Gamma) \longrightarrow \prod_{j\in\mathbb{N}} H^{2j}(X/\Gamma;\mathbb{C}) \tag{3.7}$$

In (3.7) $\mathrm{ch}_\Gamma : K_\Gamma^0(X) \longrightarrow H^0(X,\Gamma)$ is as in §1 above. $H^0(X,\Gamma) \longrightarrow \prod_{j\in\mathbb{N}} H^{2j}(X/\Gamma;\mathbb{C})$ is the projection of $H^0(X,\Gamma) = \left[\bigoplus_{\gamma\in\Gamma} \prod_{j\in\mathbb{N}} H^{2j}(X^\gamma;\mathbb{C})\right]^\Gamma$ onto the direct summand corresponding to the identity element of Γ. According to (1.19) $\mathrm{ch}_\Gamma : K_\Gamma^0(X) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow H^0(X,\Gamma)$ is an isomorphism. Therefore the traditional homotopy quotient Chern character gives for compact X a map

$$K_\Gamma^0(X) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow \left[\prod_{j\in\mathbb{N}} H^{2j}(X;\mathbb{C})\right]^\Gamma \tag{3.8}$$

which is always surjective, but fails to be injective whenever the action of Γ on X is not free. In fact, the map (3.5) is just the ordinary non-equivariant Chern character plus the observation that for a Γ-vector-bundle F on X $\mathrm{ch}(F)$ will be in $\left[\prod_{j\in\mathbb{N}} H^{2j}(X;\mathbb{C})\right]^\Gamma$.

Since $\prod_{j\in\mathbb{N}} H^{2j}(X\times_\Gamma E\Gamma;\mathbb{C}) = \prod_{j\in\mathbb{N}} H^{2j}(X/\Gamma;\mathbb{C})$ is the direct summand of $H^0(X,\Gamma)$ corresponding to the identity element of Γ, one could call $H^i(X,\Gamma)$ the "delocalized" equivariant cohomology of X [6].

§4. Γ finite: Integration over the fiber

In §4, Γ is a finite group and X is a C^∞ manifold. X is Hausdorff, finite dimensional, second countable, and without boundary. Γ acts on X by diffeomorphisms.

(4.1) *Lemma*. For $\gamma \in \Gamma$, X^γ is a C^∞ sub-manifold of X.

Proof. Choose a Γ-invariant Riemannian metric for X. X^γ is then a totally geodesic sub-manifold. Q.E.D.

Let W be another C^∞ manifold with a given C^∞ action of Γ. Assume given a Γ-equivariant C^∞ submersion ρ mapping W onto X.

$$\rho : W \longrightarrow X \tag{4.2}$$

At each $w \in W$ the derivative map $\rho' : T_w W \longrightarrow T_{\rho w} W$ is surjective. Let τ_w be the kernel of this map, so there is an exact sequence of $\mathbb{R}$ vector-spaces.

$$0 \longrightarrow \tau_w \longrightarrow T_w W \longrightarrow T_{\rho w} X \longrightarrow 0 \tag{4.3}$$

The dual of (4.3) is:

$$0 \longrightarrow T^*_{\rho w} X \longrightarrow T^*_w W \longrightarrow \tau^*_w \longrightarrow 0 \tag{4.4}$$

$\tau^* = \bigcup_{w \in W} \tau^*_w$ is a Γ-equivariant $\mathbb{R}$ vector-bundle on W. π denotes the projection $\tau^* \longrightarrow W$. τ^* is itself a C^∞ manifold acted on by Γ.

With $i = 0,1$ integration over the fiber gives a map $H^i(\tau^*,\Gamma) \longrightarrow H^i(X,\Gamma)$. To define this map we need

(4.5) *Lemma*. Let $\gamma \in \Gamma$, and let $w \in W^\gamma$. Then the derivative map $\rho' : T_w(W^\gamma) \longrightarrow T_{\rho w}(X^\gamma)$ is surjective.

Proof. Given $v \in T_{\rho w}(X^\gamma)$, choose a Γ-invariant Riemannian metric for W. There is then a unique $\tilde{v} \in T_w W$ with

$$\tilde{v} \in \tau_w^\perp \tag{4.6}$$

$$\rho'(\tilde{v}) = v \tag{4.7}$$

$\tilde{v}\gamma = \tilde{v}$, so the geodesic emanating from w with initial velocity vector $\tilde{v}$ is fixed by γ. This implies $\tilde{v} \in T_w(W^\gamma)$ and the lemma is proved. Q.E.D.

(4.8) *Lemma*. Let $\gamma \in \Gamma$. Then $\rho\pi : (\tau^*)^\gamma \longrightarrow X^\gamma$ is a submersion with oriented even-dimensional fibers.

Proof. If $x \in X^\gamma$, set $\rho_\gamma^{-1}x = \rho^{-1}x \cap W^\gamma$. Lemma (4.5) implies that $\rho_\gamma^{-1}x$ is a C^∞ sub-manifold of W^γ. Moreover, $(\rho\pi)^{-1}x \cap (\tau^*)^\gamma = T^*(\rho_\gamma^{-1}x)$. Since the cotangent bundle of any manifold is an almost complex manifold, the lemma is proved.

Q.E.D.

Integration over the fiber (e.g. see [12]) now gives maps

$$\bigoplus_{j\in\mathbb{N}} H_c^{2j}(\tau^{*\gamma};\mathbb{C}) \longrightarrow \bigoplus_{j\in\mathbb{N}} H_c^{2j}(X^\gamma;\mathbb{C})$$

$$\bigoplus_{j\in\mathbb{N}} H_c^{2j+1}(\tau^{*\gamma};\mathbb{C}) \longrightarrow \bigoplus_{j\in\mathbb{N}} H_c^{2j+1}(X^\gamma;\mathbb{C})$$

Taking the direct sum over $\gamma \in \Gamma$, then yields the desired map

$$(\rho\pi)_* : H^i(\tau^*,\Gamma) \longrightarrow H^i(X,\Gamma) \tag{4.9}$$

§5. Γ finite: $\mathrm{Td}(\tau^*,\Gamma)$

Γ, X, $\rho : W \longrightarrow X$, τ, τ^* are as in §4.

Γ_γ denotes the subgroup of Γ generated by γ. Suppose given on W^γ a Γ_γ-vector-bundle F. For $w \in W^\gamma$, let $\lambda_1,\lambda_2,\ldots,\lambda_r$ be the (distinct) eigen-values of $\gamma : F_w \longrightarrow F_w$. F is the direct sum

$$F = F^1 \oplus F^2 \oplus \ldots \oplus F^r \tag{5.1}$$

where the action of γ on F^i is multiplication by λ_i. Define $ch^\gamma(F) \in \bigoplus_{j\in\mathbb{N}} H^{2j}(W^\gamma;\mathbb{C})$ by:

$$ch^\gamma(F) = \sum_{i=1}^{r} \lambda_i ch(F^i) \tag{5.2}$$

In (5.2) $ch(F^i)$ is the ordinary non-equivariant Chern character of F^i.

Let $\Lambda^j F$ be the j-th exterior power of F and set

$$\mathrm{ch}^{\gamma}\lambda_{-1}F = \sum_{j=0}^{\ell} (-1)^j \mathrm{ch}^{\gamma}(\Lambda^j F) \tag{5.3}$$

In (5.3) $\ell = \dim_{\mathbb{C}}(F_w)$. Note that $\Lambda^0 F$ is the trivial line bundle $W^{\gamma} \times \mathbb{C}$ with γ acting trivially.

(5.4) *Lemma.* $\mathrm{ch}^{\gamma}\lambda_{-1}F$ is an invertible element of the ring $\bigoplus_{j\in\mathbb{N}} H^{2j}(W^{\gamma};\mathbb{C})$ if and only if none of the eigen-values $\lambda_1,\lambda_2,\ldots,\lambda_r$ is 1.

Proof. Let m_i be the multiplicity of λ_i. The zero-dimensional component of $\mathrm{ch}^{\gamma}\lambda_{-1}F$ is $\prod_{i=1}^{r} (1-\lambda_i)^{m_i}$. Q.E.D.

If $\gamma \in \Gamma$ and $w \in W^{\gamma}$, then τ_w and $T_w(W^{\gamma})$ are both contained in $T_w(W)$. Define Γ_{γ}-equivariant $\mathbb{R}$ vector-bundles θ, ν on W^{γ} by

$$\theta_w = \tau_w \cap T_w(W^{\gamma}) \tag{5.5}$$

$$\nu_w = \tau_w / \tau_w \cap T_w(W^{\gamma}) \tag{5.6}$$

The action of γ on θ is trivial, but the action of γ on ν is quite non-trivial. In fact (5.4) applies to $\nu \otimes_{\mathbb{R}} \mathbb{C}$, so $\mathrm{ch}^{\gamma}\lambda_{-1}(\nu\otimes_{\mathbb{R}}\mathbb{C})$ is an invertible element of $\bigoplus_{j\in\mathbb{N}} H^{2j}(W^{\gamma};\mathbb{C})$.

Define $\mathrm{Td}(\tau^*,\gamma) \in \prod_{j\in\mathbb{N}} H^{2j}(\tau^{*\gamma};\mathbb{C})$ by:

$$\mathrm{Td}(\tau^*,\gamma) = \pi^*\left[\frac{\mathrm{Td}(\theta\otimes_{\mathbb{R}}\mathbb{C})}{\mathrm{ch}^{\gamma}\lambda_{-1}(\nu\otimes_{\mathbb{R}}\mathbb{C})}\right] \tag{5.7}$$

In (5.7), $\mathrm{Td}(\theta \otimes_{\mathbb{R}} \mathbb{C})$ is the usual non-equivariant Todd class of $\theta \otimes_{\mathbb{R}} \mathbb{C}$. $\pi^* : H^*(W^\gamma;\mathbb{C}) \longrightarrow H^*(\tau^{*\gamma};\mathbb{C})$ is the map of cohomology induced by $\pi : \tau^{*\gamma} \longrightarrow W^\gamma$.

$\mathrm{Td}(\tau^*,\Gamma)$ is defined:

$$\mathrm{Td}(\tau^*,\Gamma) = \bigoplus_{\gamma\in\Gamma} \mathrm{Td}(\tau^*,\gamma) \tag{5.8}$$

$$\mathrm{Td}(\tau^*,\Gamma) \in \left[\bigoplus_{\gamma\in\Gamma} \bigoplus_{j\in\mathbb{N}} H^{2j}(\tau^{*\gamma};\mathbb{C})\right]^\Gamma$$

§6. Γ finite: Index Theorem

Γ, X, $\rho : W \longrightarrow X$, τ, τ^* are as in §4.

Let E^0, E^1 be C^∞ Γ-vector-bundles on W. $C^\infty(E^i)$ is the vector space of all C^∞ sections of E^i. Let $D : C^\infty(E^0) \longrightarrow C^\infty(E^1)$ be a pseudo-differential operator such that:

D is Γ-equivariant (6.1)

For each $x \in X$, D restricts to $\rho^{-1}x$ to give an elliptic pseudo-differential operator $D_x : C^\infty(E^0|\rho^{-1}x) \longrightarrow C^\infty(E^1|\rho^{-1}x)$. (6.2)

D is trivial at infinity. (6.3)

In brief, D is a Γ-equivariant family of elliptic pseudo-differential operators.

(6.3) asserts that each operator D_x is trivial at infinity, and that there exists a compact set Λ in X with D_x trivial for $x \notin \Lambda$. If W is compact, then (6.3) is automatically satisfied.

The index of D is an element of $K_\Gamma^0(X)$. For example, suppose that W is compact. Assume also that $\dim_{\mathbb{C}}[\text{Kernel } D_x]$ and $\dim_{\mathbb{C}}[\text{Cokernel } D_x]$ are locally constant functions on X. Since $\rho : W \longrightarrow X$ is surjective, X is compact and Kernel D, Cokernel D are Γ-vector-bundles on X. Then in this case

$$\text{Index } (D) = \text{Kernel } D - \text{Cokernel } D \tag{6.4}$$

More generally, W is not compact and $\dim_{\mathbb{C}}[\text{Kernel } D_x]$, $\dim_{\mathbb{C}}[\text{Cokernel } D_x]$ are not locally constant functions on X. Thus there is a difficulty in defining Index $(D) \in K_\Gamma^0(X)$. This problem can be overcome in various ways. See [1,5]. One very pleasant way is to use Kasparov KK theory [23] and the Green-Julg theorem [18,22]. In this approach, Sobolev spaces are used to quite directly construct from D an element of $KK(\mathbb{C}, C_0(X) \times \Gamma)$. Here $C_0(X)$ is the abelian C^* algebra of all continuous complex-valued functions on X which vanish at infinity. $C_0(X) \times \Gamma$ is the crossed-product C^* algebra arising from the action of Γ on $C_0(X)$. The Green-Julg theorem gives an isomorphism $KK(\mathbb{C}, C_0(X) \times \Gamma) \cong K_\Gamma^0(X)$ so starting from D we obtain Index $(D) \in K_\Gamma^0(X)$. Applying $ch_\Gamma : K_\Gamma^0(X) \longrightarrow H^0(X, \Gamma)$ yields $ch_\Gamma(\text{Index } D) \in H^0(X, \Gamma)$.

The symbol σ of D is a map of Γ-vector-bundles on τ^*:

$$\sigma : \pi^* E^0 \longrightarrow \pi^* E^1 \tag{6.5}$$

$\pi^{*}E^{i}$ is the pull-back via π of E^{i}. By definition the support of σ is the set of all $v \in \tau^{*}$ such that $\sigma(v) : E^{0}_{\pi v} \longrightarrow E^{1}_{\pi v}$ is not an isomorphism of $\mathbb{C}$ vector-spaces. (6.3) implies that the support of σ is a compact subset of τ^{*}. Hence σ determines an element in $K^{0}_{\Gamma}(\tau^{*})$, and applying $ch_{\Gamma} : K^{0}_{\Gamma}(\tau^{*}) \longrightarrow H^{0}(\tau^{*},\Gamma)$ we obtain $ch_{\Gamma}(\sigma) \in H^{0}(\tau^{*},\Gamma)$.

The usual cup product gives a pairing

$$H^{i}_{c}(\tau^{*};\mathbb{C}) \underset{\mathbb{C}}{\otimes} H^{j}(\tau^{*\gamma};\mathbb{C}) \longrightarrow H^{i+j}_{c}(\tau^{*\gamma};\mathbb{C}) \tag{6.6}$$

Using this pairing form $ch(\sigma) \cup Td(\tau^{*},\Gamma)$.

$$ch(\sigma) \cup Td(\tau^{*},\Gamma) \in H^{0}(\tau^{*},\Gamma) \tag{6.7}$$

With $(\rho\pi)_{*} : H^{0}(\tau^{*},\Gamma) \longrightarrow H^{0}(X,\Gamma)$ as in (4.9) we then have:

(6.8) *Theorem* (Atiyah-Singer [4,5]).

$$ch_{\Gamma}(\text{Index } D) = (\rho\pi)_{*}(ch(\sigma) \cup Td(\tau^{*},\Gamma))$$

Proof. See [4,5].

§7. Γ countable: Chern character for proper actions

X is a C^∞ manifold. X is Hausdorff, finite dimensional, second countable, and without boundary. Γ is a countable discrete group acting on X by a C^∞ (right) action $X \times \Gamma \longrightarrow X$.

Form the reduced crossed-product C^* algebra $C_0(X) \rtimes \Gamma$. See Appendix 1 below for a precise definition of this C^* algebra. Define the equivariant K theory $K^i_\Gamma(X)$ to be the K theory of this C^* algebra.

(7.1) *Definition.* $K^i_\Gamma(X) = K_i[C_0(X) \rtimes \Gamma] \quad i = 0,1.$

[10] is an excellent reference for C^* algebra K theory. In the Kasparov notation [23], we are defining $K^i_\Gamma(X)$ to be $KK^i(\mathbb{C}, C_0(X) \rtimes \Gamma)$.

(7.2) *Definition.* The action of Γ on X is *proper* if the map $X \times \Gamma \longrightarrow X \times X$ which takes (x,γ) to $(x,x\gamma)$ is proper.

For a proper C^∞ action each isotropy group is finite, the Palais slice theorem [26] is valid, and the quotient space X/Γ is Hausdorff and is an orbifold.

Any action of a finite group is proper. If Γ is finite, then the Green-Julg theorem [18,22] asserts that $K^i_\Gamma(X)$ as defined by Atiyah and Segal [4,30] is isomorphic to $K_i[C_0(X) \rtimes \Gamma]$.

For the remainder of §7 assume that the action of Γ on X is C^∞ and proper.

Set $\hat{X} = \{(x,\gamma) \in X \times \Gamma \mid x\gamma = x\}$.

Γ acts on $\hat{X}$ by $(x,\gamma)\alpha = (x\alpha, \alpha^{-1}\gamma\alpha)$. $\hat{X}$ is a C^{∞} sub-manifold of $X \times \Gamma$. The action of Γ on $\hat{X}$ is proper so $\hat{X}/\Gamma$ is an orbifold. Define the equivariant cohomology $H^i(X,\Gamma)$ $i = 0,1$ by:

$$
\begin{aligned}
H^0(X,\Gamma) &= \bigoplus_{j\in\mathbb{N}} H_c^{2j}(\hat{X}/\Gamma;\mathbb{C}) \\
H^1(X,\Gamma) &= \bigoplus_{j\in\mathbb{N}} H_c^{2j+1}(\hat{X}/\Gamma;\mathbb{C})
\end{aligned}
\tag{7.3}
$$

The Chern character $ch_\Gamma : K_\Gamma^i(X) \longrightarrow H^i(X,\Gamma)$ is defined as follows. $C_c^{\infty}(X)$ denotes the compactly supported C^{∞} complex-valued functions on X. If $f \in C_c^{\infty}(X)$ and $\gamma \in \Gamma$, then f^{γ} is:

$$f^{\gamma}(x) = f(x\gamma) \tag{7.4}$$

Consider the algebra $C_c^{\infty}(X,\Gamma)$ whose elements are all finite formal sums $\sum_{\gamma\in\Gamma} f_\gamma[\gamma]$ where $f_\gamma \in C_c^{\infty}(X)$ and $\gamma \in \Gamma$. Addition and multiplication in $C_c^{\infty}(X,\Gamma)$ are

$$\left[\sum_{\gamma\in\Gamma} f_\gamma[\gamma]\right] + \left[\sum_{\gamma\in\Gamma} h_\gamma[\gamma]\right] = \sum_{\gamma\in\Gamma} (f_\gamma + h_\gamma)[\gamma]$$

$$(f_\gamma[\gamma])(h_\alpha[\alpha]) = f_\gamma h_\alpha^{\gamma}[\gamma\alpha]$$

(7.5) *Lemma*. The inclusion $C_c^{\infty}(X,\Gamma) \subset C_0(X) \rtimes \Gamma$ induces an isomorphism $K_0[C_c^{\infty}(X,\Gamma)] \cong K_0[C_0(X) \rtimes \Gamma]$.

Proof. $C_c^{\infty}(X,\Gamma)$ is a dense sub-algebra of $C_0(X) \rtimes \Gamma$ which is closed under holomorphic functional calculus. Q.E.D.

Remarks. See Appendix 1 below for a detailed proof that $C_c^\infty(X,\Gamma)$ is closed under holomorphic functional calculus. Lemma (7.5) is not valid when the action of Γ on X is not proper.

Next, let us take the cyclic cohomology of $C_c^\infty(X,\Gamma)$. We use the notation and conventions of [16]. As a topological vector space $C_c^\infty(X,\Gamma) = C_c^\infty(X\times\Gamma)$. The evident isomorphism is:

$$\left[\sum_{\gamma\in\Gamma} f_\gamma[\gamma]\right](x,\alpha) = f_\alpha(x) \tag{7.6}$$

$C_c^\infty(X\times\Gamma)$ is topologized by the C^∞ topology, and thus $C_c^\infty(X,\Gamma)$ is topologized. Following [16] the cyclic cohomology groups $H^{ev}(C_c^\infty(X,\Gamma))$ and $H^{odd}(C_c^\infty(X,\Gamma))$ are taken with this topology on $C_c^\infty(X,\Gamma)$.

To identify $H^{ev}(C_c^\infty(X,\Gamma))$ and $H^{odd}(C_c^\infty(X,\Gamma))$ in familiar topological terms, let $\Omega_j^\Gamma(\hat{X})$ be the vector-space of all j-dimensional de Rham currents on $\hat{X}$ which are fixed by Γ. The de Rham complex

$$0 \longleftarrow \Omega_0^\Gamma(\hat{X}) \xleftarrow{\partial} \Omega_1^\Gamma(\hat{X}) \xleftarrow{\partial} \cdots \xleftarrow{\partial} \Omega_n^\Gamma(\hat{X}) \longleftarrow 0$$

has for its j-th homology $H_j^\infty(\hat{X}/\Gamma;\mathbb{C})$, the j-th homology group of $\hat{X}/\Gamma$ using countable locally finite chains with coefficients $\mathbb{C}$. This is isomorphic to the j-th Borel-Moore [11] homology of $\hat{X}/\Gamma$ with unrestricted supports and coefficients $\mathbb{C}$.

$\Omega_c^i(X)$ denotes the compactly supported C^∞ i-forms on X. If $\omega \in \Omega_c^i(X)$ and $\gamma \in \Gamma$, then ω^γ is:

$$\omega^\gamma(v_1,v_2,\ldots,v_i) = \omega(v_1\gamma,v_2\gamma,\ldots,v_i\gamma) \tag{7.7}$$

$$v_1, \ldots, v_i \in T_x X$$

$\Omega_c^i(X,\Gamma)$ denotes the vector-space whose elements are all finite formal sums $\sum_{\gamma\in\Gamma} \omega_\gamma[\gamma]$ with $\omega_\gamma \in \Omega_c^i(X)$. $d : \Omega_c^i(X,\Gamma) \longrightarrow \Omega_c^{i+1}(X,\Gamma)$ is:

$$d\left[\sum_{\gamma\in\Gamma} \omega_\gamma[\gamma]\right] = \sum_{\gamma\in\Gamma} (d\omega_\gamma)[\gamma] \tag{7.8}$$

Addition in $\Omega_c^i(X,\Gamma)$ and multiplication $\Omega_c^i(X,\Gamma) \times \Omega_c^j(X,\Gamma) \longrightarrow \Omega_c^{i+j}(X,\Gamma)$ are:

$$\left[\sum_{\gamma\in\Gamma} \omega_\gamma[\gamma]\right] + \left[\sum_{\gamma\in\Gamma} \eta_\gamma[\gamma]\right] = \sum_{\gamma\in\Gamma} (\omega_\gamma+\eta_\gamma)[\gamma] \tag{7.9}$$

$$(\omega_\gamma[\gamma])(\eta_\alpha[\alpha]) = (\omega_\gamma \wedge \eta_\alpha^\gamma)[\gamma\alpha] \tag{7.10}$$

As a vector-space $\Omega_c^i(X,\Gamma) = \Omega_c^i(X\times\Gamma)$. The evident isomorphism is:

$$\left[\sum_{\gamma\in\Gamma} \omega_\gamma[\gamma]\right](v_1,v_2,\ldots,v_i) = \omega_\gamma(v_1,v_2,\ldots,v_i) \tag{7.11}$$

$$v_1,v_2,\ldots,v_i \in T_{(x,\gamma)}(X\times\Gamma)$$

Let Z be a j-dimensional closed de Rham current on $\hat{X}$ with Z fixed by Γ. $\varphi_Z \in H_\lambda^j(C_c^\infty(X,\Gamma))$ is:

$$\varphi_Z(a_0,a_1,\ldots,a_j) = \int_Z a_0 da_1 da_2 \ldots da_j \tag{7.12}$$

$$a_0,a_1,\ldots,a_j \in C_c^\infty(X,\Gamma)$$

(7.13) *Remark.* In (7.12) d and multiplication are as in (7.8) and (7.10). The integration over Z is done via the isomorphism of vector-spaces $\Omega_c^j(X,\Gamma) = \Omega_c^j(X\times\Gamma)$ and the inclusion $\hat{X} \subset X \times \Gamma$.

Set $\Phi(Z) = \varphi_Z$

(7.14) *Theorem.*

$$\Phi : \bigoplus_{j\in\mathbb{N}} H_{2j}^{\infty}(\hat{X}/\Gamma;\mathbb{C}) \longrightarrow H^{ev}(C_c^{\infty}(X,\Gamma))$$

$$\Phi : \bigoplus_{j\in\mathbb{N}} H_{2j+1}^{\infty}(\hat{X}/\Gamma;\mathbb{C}) \longrightarrow H^{odd}(C_c^{\infty}(X,\Gamma))$$

are isomorphisms.

Proof. Let D be the bicomplex $D^{n,m} = \Omega_{n-m}^{\Gamma}(\hat{X})$. Thus $D^{n,m}$ is all de Rham currents of dimension $n-m$ on $\hat{X}$ which are fixed by Γ. The two coboundaries in D are

$$d_1 : D^{n,m} \longrightarrow D^{n+1,m} \quad d_1 = 0$$

$$d_2 : D^{n,m} \longrightarrow D^{n,m+1} \quad d_2 = \partial$$

$\mathcal{A}$ denotes $C_c^{\infty}(X,\Gamma)$ with a unit adjoined. Let C be the (b,B) bicomplex for the (topologized) algebra $\mathcal{A}$. Thus $C^{n,m} = C^{n-m}(\mathcal{A},\mathcal{A}^*)$, the $\mathbb{C}$ vector space of all continuous $n-m+1$ linear forms on $\mathcal{A}$. The two coboundaries are

$$d_1 : C^{n,m} \longrightarrow C^{n+1,m} \quad d_1 = b, \text{ the Hochschild coboundary}$$

$$d_2 : C^{n,m} \longrightarrow C^{n,m+1} \quad d_2 = B$$

Map D to C by $\Phi : D \longrightarrow C$ where

$$\Phi(Z)(a_0, a_1, \ldots, a_j) = \int_Z a_0 da_1 da_2 \ldots da_j .$$

Here we require that the unit 1 of $\mathbb{Q}$ acts as the identity on $\Omega^i_c(X, \Gamma)$ and that $d(1) = 0$. Φ is a map of bicomplexes. Filter D and C by $F^q D = \sum_{m \geqq q} D^{n,m}$, $F^q C = \sum_{m \geqq q} C^{n,m}$. Consider the resulting spectral sequences. Using the method of [16] it can be shown that at the E^1 level Φ is an isomorphism. This implies that at the E^2 level Φ is an isomorphism. Since the spectral sequence for D has $E^2 = E^\infty$, this proves the theorem. Q.E.D.

Set $H^\infty_{ev}(\hat{X}/\Gamma;\mathbb{C}) = \bigoplus_{j \in \mathbb{N}} H^\infty_{2j}(\hat{X}/\Gamma;\mathbb{C})$. The pairing of [16]:

$$K_0(C^\infty_c(X,\Gamma)) \times H^{ev}(C^\infty_c(X,\Gamma)) \longrightarrow \mathbb{C} \tag{7.15}$$

combines with (7.5) and (7.14) to become a pairing

$$K^0_\Gamma(X) \times H^\infty_{ev}(\hat{X}/\Gamma,\mathbb{C}) \longrightarrow \mathbb{C} \tag{7.16}$$

This pairing can be interpreted as a map

$$K^0_\Gamma(X) \longrightarrow H^\infty_{ev}(\hat{X}/\Gamma,\mathbb{C}) \longrightarrow \mathbb{C} \tag{7.17}$$

where $H^\infty_{ev}(\hat{X}/\Gamma,\mathbb{C})^*$ is the dual vector-space of $H^\infty_{ev}(\hat{X}/\Gamma,\mathbb{C})$.

If $\dim_{\mathbb{C}} H^{\infty}_{ev}(\hat{X}/\Gamma,\mathbb{C}) < \infty$, then $H^{\infty}_{ev}(\hat{X}/\Gamma,\mathbb{C})^{*} = \bigoplus_{j\in\mathbb{N}} H^{2j}_{c}(\hat{X}/\Gamma;\mathbb{C})$ and (7.17) becomes

$$K^{0}_{\Gamma}(X) \longrightarrow H^{0}(X,\Gamma) \tag{7.18}$$

If $H^{\infty}_{ev}(\hat{X}/\Gamma,\mathbb{C})$ is not finite dimensional, then we can choose open sets $X_1 \subset X_2 \subset X_3 \subset \ldots$ in X such that:

$$\Gamma \text{ maps } X_i \text{ to itself.} \tag{7.19}$$

$$\dim_{\mathbb{C}} H^{\infty}_{ev}(\hat{X}/\Gamma,\mathbb{C}) < \infty \tag{7.20}$$

$$\bigcup_{i=1}^{\infty} X_i = X$$

(7.18) then applies to each X_i to give

$$K^{0}_{\Gamma}(X_i) \longrightarrow H^{0}(X_i,\Gamma) \tag{7.22}$$

But $K^{0}_{\Gamma}(X) = \underrightarrow{\text{limit}}\, K^{0}_{\Gamma}(X_i)$ and $H^{0}(X,\Gamma) = \underrightarrow{\text{limit}}\, H^{0}(X_i,\Gamma)$. Therefore passing to the limit we obtain

$$ch_{\Gamma} : K^{0}_{\Gamma}(X) \longrightarrow H^{0}(X,\Gamma) \tag{7.23}$$

which is the desired map ch_{Γ}.

Let Γ act on $X \times \mathbb{R}$ by $(x,t)\gamma = (x\gamma,t)$. $ch_{\Gamma} : K^{1}_{\Gamma}(X\times\mathbb{R}) \longrightarrow H^{1}(X,\Gamma)$ is defined by requiring commutativity in the diagram

$$
\begin{array}{ccc}
K^0_\Gamma(X\times\mathbb{R}) & \xrightarrow{\ ch_\Gamma\ } & H^0(X\times\mathbb{R},\Gamma) \\
\downarrow & & \downarrow \\
K^1_\Gamma(X) & \xrightarrow{\ ch_\Gamma\ } & H^1(X,\Gamma)
\end{array}
\tag{7.24}
$$

In (7.24) the vertical arrows are the standard "integration over the fiber" isomorphisms.

(7.25) *Theorem*. For $i = 0,1$

$$ch_\Gamma : K^i_\Gamma(X) \underset{\mathbb{Z}}{\otimes} \mathbb{C} \longrightarrow H^i(X,\Gamma)$$

is an isomorphism of $\mathbb{C}$ vector-spaces.

Proof. The two agree locally, and both have a Mayer-Vietoris exact sequence. The theorem is now proved by an induction argument.

Q.E.D.

§8. Proper actions: Integration over the fiber

Γ, X are as in §7. The action of Γ on X is C^∞ and proper. This implies that a Γ-invariant Riemannian metric can be chosen for X. Each X^γ is, therefore, a C^∞ sub-manifold of X.

Let W be another C^∞ manifold with a given C^∞ action of Γ. Assume given a Γ-equivariant C^∞ submersion ρ mapping W onto X.

$$\rho : W \longrightarrow X \tag{8.1}$$

It is not difficult to prove that the action of Γ on W must be proper.

τ, τ^*, and $\pi : \tau^* \longrightarrow W$ are as in (4.3) and (4.4). With $H^i(X,\Gamma)$ as in (7.3) there is an integration over the fibre map:

$$(\rho\pi)_* : H^i(\tau^*,\Gamma) \longrightarrow H^i(X,\Gamma) \tag{8.2}$$

To describe $(\rho\pi)_*$ we need

(8.3) *Definition.* A subset Λ of X is Γ-compact if the image of Λ in X/Γ is compact.

Denote by $\Omega^i_\Gamma(X)$ the vector-space of all C^∞ i-forms ω on X such that:

$$\omega \text{ is } \Gamma\text{-invariant.} \tag{8.4}$$

$$\text{support } (\omega) \text{ is } \Gamma\text{-compact.} \tag{8.5}$$

The de Rham complex

$$0 \longrightarrow \Omega^0_\Gamma(X) \xrightarrow{d} \Omega^1_\Gamma(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_\Gamma(X) \longrightarrow 0$$

has for its j-th homology $H^j_c(X/\Gamma;\mathbb{C})$.

Lemma (4.8) is valid in the present context, so for each $\gamma \in \Gamma$, $\rho\pi : \tau^{*\gamma} \longrightarrow X^\gamma$ is a submersion with oriented even-dimensional fibers. By definition $\hat{\tau}^* \subset \tau^* \times \Gamma$ and $\hat{X} \subset X \times \Gamma$. Map $\hat{\tau}^*$ to $\hat{X}$ by $\rho\pi \times 1$ where

$$(\rho\pi\times 1)(v,\gamma) = (\rho\pi v,\gamma) \tag{8.6}$$

Then $\rho\pi \times 1 : \hat{\tau}^* \longrightarrow \hat{X}$ is a submersion with oriented even-dimensional fibers.

The key point for (8.2) is:

(8.7) *Lemma*. Let Δ be a Γ-compact subset of $\hat{\tau}^*$. Then for each $(x,\gamma) \in \hat{X}$, $\Delta \cap (\rho\pi\times 1)^{-1}(x,\gamma)$ is compact.

Proof. Let $I_{(x,\gamma)}$ be the isotropy group of (x,γ). The image of $\Delta \cap (\rho\pi\times 1)^{-1}(x,\gamma)$ in $\hat{\tau}^*/\Gamma$ must be compact. But this image is equal to $\Delta \cap (\rho\pi\times 1)^{-1}(x,\gamma)/I_{(x,\gamma)}$. Since $I_{(x,\gamma)}$ is finite the lemma is proved. Q.E.D.

The usual integration over the fiber of differential forms [12] now gives a map of de Rham complexes $\Omega^*_\Gamma(\hat{\tau}^*) \longrightarrow \Omega^*_\Gamma(\hat{X})$ and this yields the desired map (8.2).

§9. Proper actions: $Td(\tau^*,\Gamma)$

Γ, X, $\rho : W \longrightarrow X$, τ, τ^* are as in §8.

$Td(\tau^*,\Gamma) \in \bigoplus_{j\in\mathbb{N}} H^{2j}(\hat{\tau}^*/\Gamma;\mathbb{C})$ is defined essentially as in §5, with certain nuances which are indicated below.

$\tilde{\Omega}^i_\Gamma(X)$ is the vector-space of all Γ-invariant C^∞ i-forms on X. The de Rham complex

$$0 \longrightarrow \tilde{\Omega}^0_\Gamma(X) \xrightarrow{d} \tilde{\Omega}^1_\Gamma(X) \xrightarrow{d} \dots \xrightarrow{d} \tilde{\Omega}^n_\Gamma(X) \longrightarrow 0$$

has for its j-th homology $H^j(X/\Gamma;\mathbb{C})$.

$Z(\gamma)$ denotes the centralizer of γ.

$$Z(\gamma) = \{\alpha \in \Gamma \mid \alpha^{-1}\gamma\alpha = \gamma\} \tag{9.1}$$

Note that $Z(\gamma)$ acts on W^γ. Suppose given on W^γ a $Z(\gamma)$-vector-bundle F. For $w \in W$ let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the (distinct) eigenvalues of $\gamma : F_w \longrightarrow F_w$. As a $Z(\gamma)$-vector-bundle F is then the direct sum

$$F = F^1 \oplus F^2 \oplus \ldots \oplus F^r \tag{9.2}$$

where the action of γ on F^i is multiplication by λ_i. Define $ch^\gamma(F) \in \bigoplus_{j \in \mathbb{N}} H^{2j}(W^\gamma/Z(\gamma);\mathbb{C})$ by

$$ch^\gamma(F) = \sum_{i=1}^{r} \lambda_i ch(F^i) \tag{9.3}$$

In (9.3) $ch(F^i)$ is the ordinary non-equivariant Chern character of F^i, descended to $\oplus H^{2j}(W^\gamma/Z(\gamma);\mathbb{C})$. This is found as a differential form by choosing a $Z(\gamma)$-equivariant connection for F^i. The differential form for $ch(F^i)$ so obtained is $Z(\gamma)$-invariant and thus determines an element of $\bigoplus_{j \in \mathbb{N}} H^{2j}(W^\gamma/Z(\gamma);\mathbb{C})$. Set

$$ch^\gamma\lambda_{-1}F = \sum_{j=0}^{\ell} (-1)^j ch^\gamma(\Lambda^j F), \quad \text{where} \quad \ell = \dim_{\mathbb{C}}(F_w).$$

Exactly as in (5.5) and (5.6) define $Z(\gamma)$-equivariant $\mathbb{R}$ vector-bundles θ, ν on W^γ. $ch^\gamma\lambda_{-1}(\nu \otimes_{\mathbb{R}} \mathbb{C})$ is an invertible element of

$\bigoplus_{j\in\mathbb{N}} H^{2j}(W^{\gamma}/Z(\gamma);\mathbb{C})$.

Define $Td(\tau^{*},\gamma)\ \bigoplus_{j\in\mathbb{N}} H^{2j}(\tau^{*\gamma}/Z(\gamma);\mathbb{C})$ by:

$$Td(\tau^{*},\gamma) = \pi^{*}\left[\frac{Td(\theta\otimes_{\mathbb{R}}\mathbb{C})}{ch^{\gamma}\lambda_{-1}(\nu\otimes_{\mathbb{R}}\mathbb{C})}\right] \tag{9.4}$$

In (9.4) $Td(\theta\otimes_{\mathbb{R}}\mathbb{C}) \in \bigoplus_{j\in\mathbb{N}} H^{2j}(W^{\gamma}/Z(\gamma);\mathbb{C})$ is the usual non-equivariant Todd class of $\theta\otimes_{\mathbb{R}}\mathbb{C}$, descended to $\bigoplus_{j\in\mathbb{N}} H^{2j}(W^{\gamma}/Z(\gamma);\mathbb{C})$. This is found as a differential form by choosing a $Z(\gamma)$-equivariant connection for $\theta\otimes_{\mathbb{R}}\mathbb{C}$. The differential form for $Td(\theta\otimes_{\mathbb{R}}\mathbb{C})$ so obtained is $Z(\gamma)$-equivariant and thus determines an element of $\bigoplus_{j\in\mathbb{N}} H^{2j}(W^{\gamma}/Z(\gamma);\mathbb{C})$. In (9.4) $\pi^{*} : H^{*}(W^{\gamma}/Z(\gamma);\mathbb{C}) \longrightarrow H^{*}(\tau^{*\gamma}/Z(\gamma);\mathbb{C})$ is the map of cohomology induced by $\pi : \tau^{*\gamma}/Z(\gamma) \longrightarrow W^{\gamma}/Z(\gamma)$.

(9.5) *Lemma*. Let $L = \{\gamma_1,\gamma_2,\gamma_3,\ldots\}$ be elements of finite order in Γ such that any element of finite order in Γ is conjugate to one and only one of the γ_i. Then $\hat{X}/\Gamma = \bigcup_{\gamma\in L} X^{\gamma}/Z(\gamma)$.

Proof. The evident map $\bigcup_{\gamma\in L} X^{\gamma}/Z(\gamma) \longrightarrow \hat{X}/\Gamma$ is one-to-one and onto. Q.E.D.

(9.6) *Remark*. With L as in (9.5), the identity element of Γ must be an element of L. (The identity element of Γ is of finite order, and is the unique element in Γ of order one.)

Applying (9.5) to τ^{*}, we have

$$\hat{\tau}^{*}/\Gamma = \bigcup_{\gamma\in L} \tau^{*\gamma}/Z(\gamma) \tag{9.7}$$

Using (9.7) and (9.4) define $Td(\tau^*,\Gamma)$ by:

$$Td(\tau^*,\Gamma) = \prod_{\gamma\in L} Td(\tau^*,\gamma) \tag{9.8}$$

$$Td(\tau^*,\Gamma) \in \bigoplus_{j\in\mathbb{N}} H^{2j}(\hat{\tau}^*/\Gamma;\mathbb{C}).$$

§10. Proper actions: Index Theorem

Γ, X, $\rho : W \longrightarrow X$, τ, τ^* are as in §8.

Let E^0, E^1 be C^∞ Γ-vector-bundles on W. $C^\infty(E^i)$ is the vector-space of all C^∞ sections of E^i. Let $D : C^\infty(E^0) \longrightarrow C^\infty(E^1)$ be a pseudo-differential operator such that:

D is Γ-equivariant. (10.1)

For each $x \in X$, D restricts to $\rho^{-1}x$ to give an elliptic pseudo-differential operator $D_x : C^\infty(E^0|\rho^{-1}x) \longrightarrow C^\infty(E^1|\rho^{-1}x)$. (10.2)

D has Γ-compact support. (10.3)

In brief, D is a Γ-equivariant family of elliptic pseudo-differential operators.

(10.3) asserts that each operator D_x is trivial at infinity and that there exists a Γ-compact subset Λ in X with D_x trivial for $x \notin \Lambda$.

The index of D is an element of $K^0_\Gamma(X) = KK(\mathbb{C}, C_0(X) \times \Gamma)$. Applying $ch_\Gamma : K^0_\Gamma(X) \longrightarrow H^0(X,\Gamma)$ yields $ch_\Gamma(\text{Index } D) \in H^0(X,\Gamma)$.

The symbol σ of D is a map of Γ-vector-bundles on τ^*:

$$\sigma : \pi^* E^0 \longrightarrow \pi^* E^1. \tag{10.4}$$

By definition the support of σ is the set of all $v \in \tau^*$ such that $\sigma(v) : E^0_{\pi v} \longrightarrow E^1_{\pi v}$ is not an isomorphism of $\mathbb{C}$ vector-spaces. (10.3) implies that the support of σ is a Γ-compact subset of τ^*. Hence σ determines an element of $K^0_\Gamma(\tau^*) = K_0[C_0(\tau^*) \times \Gamma]$. Applying $ch_\Gamma : K^0_\Gamma(\tau^*) \longrightarrow H^0(\tau^*,\Gamma)$ we obtain $ch_\Gamma(\sigma) \in H^0(\tau^*,\Gamma)$.

The usual cup product gives a pairing

$$H^i_c(\hat{\tau}^*/\Gamma;\mathbb{C}) \underset{\mathbb{C}}{\otimes} H^j(\hat{\tau}^*/\Gamma;\mathbb{C}) \longrightarrow H^{i+j}_c(\hat{\tau}^*/\Gamma;\mathbb{C}) \tag{10.5}$$

Using this pairing form $ch(\sigma) \cup Td(\tau^*,\Gamma)$.

$$ch(\sigma) \cup Td(\tau^*,\Gamma) \in H^0(\tau^*,\Gamma) \tag{10.6}$$

With $(\rho\pi)_* : H^0(\tau^*,\Gamma) \longrightarrow H^0(X,\Gamma)$ as in (8.2) we then have:

(10.7) *Theorem*.

$$ch_\Gamma(\text{Index } D) = (\rho\pi)_*(ch(\sigma) \cup Td(\tau^*,\Gamma)).$$

§11. Twisted homology and K homology

Let Y be a Hausdorff topological space which has the homotopy type of a CW complex. Let F be an $\mathbb{R}$ vector bundle on Y. $F - \{0\}$ denotes F with the zero section deleted. The j-th homology group of Y, twisted by F, is denoted $H_j^F(Y)$ and is defined

$$H_j^F(Y) = H_j(F, F-\{0\}; \mathbb{C}) \tag{11.1}$$

Thus $H_j^F(Y)$ is the j-th singular homology group, with coefficients the complex numbers $\mathbb{C}$, of the pair $(F, F-\{0\})$. Equivalently, choose a Euclidean structure for F. Then $H_j^F(Y) = H_j(BF, SF; \mathbb{C})$ where BF, SF are the unit ball and unit sphere bundles of F.

Let V be a C^∞ manifold. Choose a Riemannian metric for V. BTV is then an almost-complex manifold with boundary STV. BTV is oriented by its almost-complex structure so there is the Poincaré duality isomorphism

$$H_c^j(BTV) \cong H_{2n-j}(BTV, STV) \qquad n = \dim(V) \tag{11.2}$$

Since $H_c^j(V) = H_c^j(BTV)$, (11.2) can be viewed as an isomorphism

$$H_c^j(V) \cong H_{2n-j}^{TV}(V) \qquad n = \dim(V) \tag{11.3}$$

Note that in (11.3) V is not required to be oriented. In particular (11.3) gives isomorphisms

$$\bigoplus_{j\in\mathbb{N}} H_c^{2j}(V) \cong \bigoplus_{j\in\mathbb{N}} H_{2j}^{TV}(V)$$

$$\bigoplus_{i\in\mathbb{N}} H_c^{2j+1}(V) \cong \bigoplus_{i\in\mathbb{N}} H_{2j+1}^{TV}(V) \tag{11.4}$$

More generally, suppose given a direct sum decomposition of $\mathbb{R}$ vector bundles $TV = E \oplus F$. Then Poincaré duality gives an isomorphism

$$H_c^j(E) \cong H_{2n-j}^F(V) \qquad n = \dim(V) \tag{11.5}$$

From (11.5) we have isomorphisms

$$\bigoplus_{j\in\mathbb{N}} H_c^{2j}(E) \cong \bigoplus_{j\in\mathbb{N}} H_{2j}^{F}(V)$$

$$\bigoplus_{j\in\mathbb{N}} H_c^{2j+1}(E) \cong \bigoplus_{j\in\mathbb{N}} H_{2j+1}^{F}(V) \tag{11.6}$$

If F is an $\mathbb{R}$ vector bundle on Y and $f : Z \longrightarrow Y$ is a continuous map, then f determines a map of twisted homology

$$f_* : H_j^{f^*F}(Z) \longrightarrow H_j^F(Y) \tag{11.7}$$

In (11.7) f^*F is the pull-back of F via f.

Quite similar remarks hold for K homology. If F is an $\mathbb{R}$ vector bundle on Y, then the j-th K homology of Y, twisted by F, is denoted $K_j^F(Y)$ and is defined

$$K_j^F(Y) = K_j(F, F-\{0\}) \qquad j = 0,1 \tag{11.8}$$

K homology is the homology theory associated to the $\mathbb{Z} \times BU$ spectrum. A concrete realization of this theory is obtained by using the K-cycle definition of [9]. If $Z \subset Y$, then a K-cycle for (Y,Z) is a triple (M,E,φ) such that:

(i) M is a compact Spin^c manifold which may have non-empty boundary.

(ii) E is a $\mathbb{C}$ vector bundle on M.

(iii) $\varphi : M \longrightarrow Y$ is a continuous map with $\varphi(\partial M) \subset Z$.

As in [9] the equivalence relation on these K cycles is the equivalence relation generated by:

(i) Bordism

(ii) Direct sum-disjoint union

(iii) Vector bundle modification.

In the six term exact sequence

$$\begin{tikzcd}
K_0(Z) \arrow[r] & K_0(Y) \arrow[r] & K_0(Y,Z) \arrow[d] \\
K_1(Y,Z) \arrow[u] & K_1(Y) \arrow[l] & K_1(Z) \arrow[l]
\end{tikzcd}$$

the boundary map $K_i(Y,Z) \longrightarrow K_{i+1}(Z)$ takes (M,E,φ) to $(\partial M, E|\partial M, \varphi|\partial M)$.

The (homology) Chern character

$$\begin{aligned}
ch &: K_0(Y,Z) \longrightarrow \bigoplus_{j\in\mathbb{N}} H_{2j}(Y,Z,\mathbb{Q}) \\
ch &: K_1(Y,Z) \longrightarrow \bigoplus_{j\in\mathbb{N}} H_{2j+1}(Y,Z,\mathbb{Q})
\end{aligned} \tag{11.9}$$

is

$$\mathrm{ch}(M,E,\varphi) = \varphi_*(\mathrm{ch}E\cup\mathrm{Td}(M)\cap[M,\partial M]) \tag{11.10}$$

In (11.10) $\varphi_* : H_*(M,\partial M,\mathbb{Q}) \longrightarrow H_*(Y,Z,\mathbb{Q})$ is the map of rational homology induced by φ. $[M,\partial M]$ is the orientation cycle of M.

§12. Improper actions: Integration over the fiber

Γ is a countable group acting by a C^∞ action on a C^∞ manifold X. The action $X \times \Gamma \longrightarrow X$ is not required to be proper. See Appendix 2 below for the case when Γ is not countable.

As in §7, $C_0(X) \rtimes \Gamma$ denotes the reduced cross-product C^* algebra arising from the action of Γ on $C_0(X)$. The equivariant K theory $K^i_\Gamma(X)$ is defined as in (7.1) by:

$$K^i_\Gamma(X) = K_i[C_0(X)\rtimes\Gamma] \qquad i = 0,1 \tag{12.1}$$

To define the equivariant cohomology $H^i(X,\Gamma)$ let $S(\Gamma) = \{\gamma \in \Gamma \mid \gamma$ is of finite order$\}$. The identity element of Γ is in $S(\Gamma)$. Set

$$\hat{X} = \{(x,\gamma) \in X \times \Gamma \mid x\gamma = x \ \text{ and } \ \gamma \in S(\Gamma)\}. \tag{12.2}$$

$\hat{X}$ is a C^∞ sub-manifold of $X \times \Gamma$. The reason for this, is that given $\gamma \in S(\Gamma)$ we can choose a Riemannian metric for X which is invariant under γ. For $\gamma \in \Gamma$ of infinite order, X^γ might not be a manifold. If the action of Γ on X is proper, then all isotropy groups are finite, so the $\hat{X}$ of (12.2) agrees with the $\hat{X}$ of (7.3). Γ acts on $\hat{X}$ by $(x,\gamma)\alpha = (x\alpha, \alpha^{-1}\gamma\alpha)$. The quotient space $\hat{X}/\Gamma$ may be non-Hausdorff and quite pathological. Thus standard algebraic topology may not apply to $\hat{X}/\Gamma$. We can, however, form the homotopy quotient $\hat{X} \times_{\Gamma} E\Gamma$. Denote the j-th homology group of $\hat{X} \times_{\Gamma} E\Gamma$, twisted by the $\mathbb{R}$ vector bundle $T\hat{X} \times_{\Gamma} E\Gamma$, by $H_j^{T\hat{X}}(\hat{X}\times_{\Gamma} E\Gamma)$. The equivariant cohomology $H^i(X,\Gamma)$ $i = 0,1$ is defined:

$$
\begin{aligned}
H^0(X,\Gamma) &= \bigoplus_{j\in\mathbb{N}} H_{2j}^{T\hat{X}}(\hat{X}\times_{\Gamma} E\Gamma) \\
H^1(X,\Gamma) &= \bigoplus_{j\in\mathbb{N}} H_{2j+1}^{T\hat{X}}(\hat{X}\times_{\Gamma} E\Gamma)
\end{aligned}
\tag{12.3}
$$

The next lemma asserts that for proper actions $H^i(X,\Gamma)$ as defined in (7.3) agrees with $H^i(X,\Gamma)$ as defined in (12.3).

(12.4) *Lemma*. If the action of Γ on X is proper, then there are isomorphisms

$$\bigoplus_{j\in\mathbb{N}} H_c^{2j}(\hat{X}/\Gamma;\mathbb{C}) \cong \bigoplus_{j\in\mathbb{N}} H_{2j}^{T\hat{X}}(\hat{X}\times_{\Gamma} E\Gamma)$$

$$\bigoplus_{j\in\mathbb{N}} H_c^{2j+1}(\hat{X}/\Gamma;\mathbb{C}) \cong \bigoplus_{j\in\mathbb{N}} H_{2j+1}^{T\hat{X}}(\hat{X}\times_{\Gamma} E\Gamma)$$

Proof. Projection on the first factor gives a map of pairs

$$(T\hat{X}\times_{\Gamma} E\Gamma, T\hat{X}\times_{\Gamma} E\Gamma-\{0\}) \longrightarrow (T\hat{X}/\Gamma, T\hat{X}-\{0\}/\Gamma). \tag{12.5}$$

The fibers of this map are classifying spaces of finite groups. Therefore with coefficients $\mathbb{C}$ the map induces a homology isomorphism.

Set $H_j^{\hat{TX}}(\hat{X}/\Gamma) = H_j(T\hat{X}/\Gamma, T\hat{X}-\{0\}/\Gamma;\mathbb{C})$. The proof is now completed by observing that in the context of rational homology manifolds (11.4) remains valid. Hence there are Poincaré duality isomorphisms:

$$\bigoplus_{j\in\mathbb{N}} H_c^{2j}(\hat{X}/\Gamma;\mathbb{C}) \cong \bigoplus_{j\in\mathbb{N}} H_{2j}^{\hat{TX}}(\hat{X}/\Gamma)$$

$$\bigoplus_{j\in\mathbb{N}} H_c^{2j+1}(\hat{X}/\Gamma;\mathbb{C}) \cong \bigoplus_{j\in\mathbb{N}} H_{2j+1}^{\hat{TX}}(\hat{X}/\Gamma) \tag{12.6}$$

Q.E.D.

Let W be a C^∞ manifold on which Γ is acting by a proper C^∞ action. Assume given a C^∞ Γ-equivariant submersion ρ mapping W onto X, τ, τ^*, π are as in (4.3) and (4.4). Integration over the fiber maps $H^i(\tau^*,\Gamma)$ to $H^i(X,\Gamma)$.

$$(\rho\pi)_* : H^i(\tau^*,\Gamma) \longrightarrow H^i(X,\Gamma) \tag{12.7}$$

The action of Γ on X is not required to be proper, so in (12.7) $H^i(X,\Gamma)$ is as in (12.3). The action of Γ on W is proper. To define $(\rho\pi)_*$ we need:

(12.8) *Lemma*. Let $H_j^{\hat{TX}}(\hat{W}\times_\Gamma E\Gamma)$ be the j-th homology of $\hat{W}\times_\Gamma E\Gamma$, twisted by $\rho^*\hat{TX}\times_\Gamma E\Gamma$. Then there are isomorphisms

$$H^0(\tau^*,\Gamma) \cong \bigoplus_{j\in\mathbb{N}} H_{2j}^{\hat{TX}}(\hat{W}\times_\Gamma E\Gamma)$$

$$H^1(\tau^*,\Gamma) \cong \bigoplus_{j\in\mathbb{N}} H_{2j+1}^{\hat{TX}}(\hat{W}\times_\Gamma E\Gamma) \tag{12.9}$$

Proof. $\rho : W \longrightarrow X$ determines a map $\hat{W} \longrightarrow \hat{X}$ which will also be denoted ρ. $\rho^*\hat{TX}$ is the pull-back via ρ of $\hat{TX}$. Set

$$H_j^{\hat{TX}}(\hat{W}/\Gamma) = H_j(\rho^*\hat{TX}/\Gamma, \rho^*\hat{TX}-\{0\}/\Gamma) \tag{12.10}$$

Projection on the first factor gives a homology isomorphism

$$H_j^{\hat{TX}}(\hat{W}\times_\Gamma E\Gamma) \longrightarrow H_j^{\hat{TX}}(\hat{W}/\Gamma) \tag{12.11}$$

Since the action of Γ on W is proper we may choose a Γ-invariant Riemannian metric for W. This gives a direct-sum decomposition of $\mathbb{R}$ Γ-vector-bundles on $\hat{W}$.

$$\hat{TW} = \hat{\tau} \oplus \rho^*\hat{TX} \tag{12.12}$$

In the context of rational homology manifolds (11.6) remains valid to give isomorphisms

$$\bigoplus_{j\in\mathbb{N}} H_c^{2j}(\hat{\tau}/\Gamma;\mathbb{C}) \cong \bigoplus_{j\in\mathbb{N}} H_{2j}^{\hat{TX}}(\hat{W}/\Gamma)$$

$$\bigoplus_{j\in\mathbb{N}} H_c^{2j+1}(\hat{\tau}/\Gamma;\mathbb{C}) \cong \bigoplus_{j\in\mathbb{N}} H_{2j+1}^{\hat{TX}}(\hat{W}/\Gamma) \tag{12.13}$$

The Γ-invariant Riemannian metric for W identifies τ and τ^{*}, so we have the evident isomorphism.

$$H^{i}(\tau^{*},\Gamma) \cong H^{i}(\tau,\Gamma) \quad i = 0,1 \tag{12.14}$$

Combining (12.11), (12.13) and (12.14) proves the lemma. Q.E.D.

(11.7) applies to give a map

$$H_{j}^{T\hat{X}}(\hat{W}\times_{\Gamma} E\Gamma) \longrightarrow H_{j}^{T\hat{X}}(\hat{X}\times_{\Gamma} E\Gamma) \tag{12.15}$$

$(\rho\pi)_{*} : H^{i}(\tau^{*},\Gamma) \longrightarrow H^{i}(X,\Gamma)$ is obtained by composing the isomorphism of (12.9) and the map of (12.15).

§13. Improper actions: $Td(\tau^{*},\Gamma)$

Γ, X, $\rho : W \longrightarrow X$, τ, τ^{*} are as in §12. The action of Γ on W is proper, but the action of Γ on X is not required to be proper.

$Td(\tau^{*},\Gamma)$ is defined as in §9. There is no change since the action of Γ on W is proper. Suppose now that $\rho_1 : W_1 \longrightarrow X$ and $\rho_2 : W_2 \longrightarrow X$ are both Γ-equivariant C^{∞} submersions. Assume that ρ_i maps W_i onto X. Let $f : W_1 \longrightarrow W_2$ be a C^{∞} Γ-equivariant map such that $\rho_1 = \rho_2 f$. According to (12.9) there are isomorphisms

$$H^0(\tau_i^*,\Gamma) \cong \bigoplus_{j\in\mathbb{N}} H_{2j}^{T\hat{X}}(\hat{W}_i \underset{\Gamma}{\times} E\Gamma)$$

$$H^1(\tau_i^*,\Gamma) \cong \bigoplus_{j\in\mathbb{N}} H_{2j+1}^{T\hat{X}}(\hat{W}_i \underset{\Gamma}{\times} E\Gamma) \tag{13.1}$$

(11.7) applies to give a map

$$H_j^{T\hat{X}}(\hat{W}_1 \underset{\Gamma}{\times} E\Gamma) \longrightarrow H_j^{T\hat{X}}(\hat{W}_2 \underset{\Gamma}{\times} E\Gamma) \tag{13.2}$$

Combining (13.1) and (13.2) yields a Gysin "wrong way" map

$$f_* : H^i(\tau_1^*,\Gamma) \longrightarrow H^i(\tau_2^*,\Gamma) \tag{13.3}$$

It is immediate from the definition of integration over the fiber that there is commutativity in the diagram

$$\begin{tikzcd} H^i(\tau_1^*,\Gamma) \arrow[rr, "f_*"] \arrow[dr, "(\rho_1\pi_1)_*"'] & & H^i(\tau_2^*,\Gamma) \arrow[dl, "(\rho_2\pi_2)_*"] \\ & H^i(X,\Gamma) & \end{tikzcd} \tag{13.4}$$

Let us now consider the K-theory versions of f_* and $(\rho\pi)_*$. These shall be denoted:

$$f_! : K_i[C_0(\tau_1^*)\rtimes\Gamma] \longrightarrow K_i[C_0(\tau_2^*)\rtimes\Gamma] \tag{13.5}$$

$$(\rho\pi)_! : K_i[C_0(\tau^*)\rtimes\Gamma] \longrightarrow K_i[C_0(X)\rtimes\Gamma] \tag{13.6}$$

To define these it is convenient to recall the definition of K-oriented map.

(13.7) *Definition.* Let V_1, V_2 be C^∞ manifolds and let $h : V_1 \longrightarrow V_2$ be a C^∞ map. h is *K-oriented* if a Spin^c structure is given for the $\mathbb{R}$ vector bundle $TV_1 \oplus h^* TV_2$.

It is well known [3] that a K-oriented map $h : V_1 \longrightarrow V_2$ induces a Gysin "wrong way" map $K^*(V_1) \longrightarrow K^*(V_2)$. From the Kasparov point of view [23], h determines an element of $KK^*(C_0(V_1), C_0(V_2))$ and this gives the Gysin map $K^*(V_1) \longrightarrow K^*(V_2)$.

If Γ acts by diffeomorphisms on V_1 and V_2 we require:

(i) The Γ action of V_1 is proper.

(ii) h is Γ-equivariant.

(iii) A Γ-equivariant Spin^c structure is given for $TV_1 \oplus h^* TV_2$.

When (i), (ii), (iii) are satisfied, h determines an element of $KK^*(C_0(V_1) \rtimes \Gamma, C_0(V_2) \rtimes \Gamma)$ and thus h induces a Gysin map $K_*[C_0(V_1) \rtimes \Gamma] \longrightarrow K_*[C_0(V_2) \rtimes \Gamma]$.

For (13.5) let $f : W_1 \longrightarrow W_2$ be as above. $f' : \tau_1 \longrightarrow \tau_2$ is the derivative of f. Since the action of Γ on W_i is proper, a Γ-invariant Riemannian metric can be chosen for W_i. This identifies τ_i with τ_i^* so f' becomes a map from τ_1^* to τ_2^*.

$$f' : \tau_1^* \longrightarrow \tau_2^* \tag{13.8}$$

One now easily checks that the Γ-equivariant $\mathbb{R}$ vector bundle $T\tau_1^* \oplus f'^* T\tau_2^*$ is of the form $E \oplus E = E \otimes_{\mathbb{R}} \mathbb{C}$ and thus is a Γ-equivariant $\mathbb{C}$

vector bundle on τ_1^*. (i), (ii), (iii) are satisfied and we have $f_!$: $K_i[C_0(\tau_1^*)\times\Gamma] \longrightarrow K_i[C_0(\tau_2^*)\times\Gamma]$.

(13.9) Remark. $f_!$ depends only on the homotopy class (as a C^∞ Γ-equivariant map with $\rho_2 f = \rho_1$) of f. If $f : W_1 \longrightarrow W_2$ and g : $W_2 \longrightarrow W_3$ are as above then $(gf)_! = g_! f_!$. Note that f is not required to be a proper map.

For (13.6) one checks easily that the Γ-equivariant $\mathbb{R}$ vector bundle $T\tau^* \oplus (\rho\pi)^* TX$ is of the form $F \oplus F = F \otimes_{\mathbb{R}} \mathbb{C}$ and thus is a Γ-equivariant $\mathbb{C}$ vector bundle on τ^*. (i), (ii), (iii) are satisfied and we have $(\rho\pi)_! : K_i[C_0(\tau^*)\times\Gamma] \longrightarrow K_i[C_0(X)\times\Gamma]$.

If $x \in X$, $(\rho\pi)^{-1}x = T^*(\rho^{-1}x)$ is an even-dimensional almost-complex manifold. Let D_x be the Dirac operator of $(\rho\pi)^{-1}x$. Then $\{D_x\}$ is a Γ-equivariant family of Dirac operators parametrized by X. Hence $\{D_x\}$ can be viewed as an element of $KK^0_\Gamma(C_0(\tau^*), C_0(X))$.

$$\{D_x\} \in KK^0_\Gamma(C_0(\tau^*), C_0(X)) \tag{13.10}$$

Applying Kasparov's map

$$KK^0_\Gamma(C_0(\tau^*), C_0(X)) \longrightarrow KK^0(C_0(\tau^*)\times\Gamma, C_0(X)\times\Gamma)$$

to $\{D_x\}$ yields the element in $KK^0(C_0(\tau^*)\times\Gamma, C_0(X)\times\Gamma)$ which gives $(\rho\pi)_! : K_i[C_0(\tau^*)\times\Gamma] \longrightarrow K_i[C_0(X)\times\Gamma]$.

Remark. There is commutativity in the diagram

$$\begin{tikzcd}
K_i[C_0(\tau_1^*)\rtimes\Gamma] \arrow[rr, "f_!"] \arrow[dr, "(\rho_1\pi_1)_!"'] & & K_i[C_0(\tau_2^*)\rtimes\Gamma] \arrow[dl, dashed, "(\rho_2\pi_2)_!"] \\
& K_i[C_0(X)\rtimes\Gamma] &
\end{tikzcd} \tag{13.11}$$

For the next proposition, let $ch_\Gamma \cup Td(\tau^*,\Gamma)$ be the map which sends $\xi \in K_i[C_0(\tau^*)\rtimes\Gamma]$ to $ch_\Gamma(\xi) \cup Td(\tau^*,\Gamma) \in H^i(\tau^*,\Gamma)$.

(13.12) *Proposition.* There is commutativity in the diagram

$$\begin{tikzcd}
K_i[C_0(\tau_1^*)\rtimes\Gamma] \arrow[r, "f_!"] \arrow[d, "ch_\Gamma \cup Td(\tau_1^*,\Gamma)"'] & K_i[C_0(\tau_2^*)\rtimes\Gamma] \arrow[d, "ch_\Gamma \cup Td(\tau_2^*,\Gamma)"] \\
H^i(\tau_1^*,\Gamma) \arrow[r, "f_*"'] & H^i(\tau_2^*,\Gamma)
\end{tikzcd}$$

§14. Improper actions. $K^i(X,\Gamma)$

Γ, X, $K^i_\Gamma(X)$, $H^i(X,\Gamma)$ are as in §12. By definition $K^i_\Gamma(X) = K_i[C_0(X)\rtimes\Gamma]$, and

$$\begin{aligned}
H^0(X,\Gamma) &= \bigoplus_{j\in\mathbb{N}} H_{2j}^{T\hat{X}}(\hat{X}\underset{\Gamma}{\times}E\Gamma) \\
H^1(X,\Gamma) &= \bigoplus_{j\in\mathbb{N}} H_{2j+1}^{T\hat{X}}(\hat{X}\underset{\Gamma}{\times}E\Gamma)
\end{aligned} \tag{14.1}$$

At the present time [17] the Chern character (as a pairing $H_i(X,\Gamma) \times$

$K_i[C_0(X)\rtimes\Gamma] \longrightarrow \mathbb{C})$ has been defined only for certain naturally arising elements of $H_i(X,\Gamma)$. So instead of constructing a Chern character $K_i[C_0(X)\rtimes\Gamma] \longrightarrow H^i(X,\Gamma)$ we shall introduce a geometric K theory $K^i(X,\Gamma)$ and a map $\mu : K^i(X,\Gamma) \longrightarrow K_i[C_0(X)\rtimes\Gamma]$. In every computed example μ is an isomorphism. We conjecture that μ is always an isomorphism. Moreover, there is a Chern character

$$ch_\Gamma : K^i(X,\Gamma) \longrightarrow H^i(X,\Gamma) \tag{14.2}$$

The picture is:

$$\begin{tikzcd} K^i(X,\Gamma) \arrow[r, "\mu"] \arrow[d, "ch_\Gamma"] & K_i[C_0(X)\rtimes\Gamma] \arrow[d] \\ H^i(X,\Gamma) \arrow[r] & K_i[C_0(X)\rtimes\Gamma] \otimes_{\mathbb{Z}} \mathbb{C} \end{tikzcd} \tag{14.3}$$

In (14.3) the right vertical arrow is the tautological map. It will turn out that $ch_\Gamma : K^i(X,\Gamma) \longrightarrow H^i(X,\Gamma)$ gives an isomorphism $K^i_\Gamma(X,\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow H^i(X,\Gamma)$. The map $H^i(X,\Gamma) \longrightarrow K_i[C_0(X)\rtimes\Gamma] \otimes_{\mathbb{Z}} \mathbb{C}$ is then defined by requiring commutativity in the diagram (14.3).

For $K^i(X,\Gamma)$ let $\mathscr{C}(X,\Gamma)$ be the category whose objects are all pairs (W,ρ) such that:

(i) W is a C^∞ manifold with a given proper action of Γ by diffeomorphisms.

(ii) ρ is a C^∞ Γ-equivariant submersion mapping W onto X.

For such a (W,ρ) τ, τ^*, π are as in (4.3) and (4.4).

Let (W_1,ρ_1) and (W_2,ρ_2) be objects in $\mathscr{C}(X,\Gamma)$. A morphism from (W_1,ρ_1) to (W_2,ρ_2) is a C^∞ Γ-equivariant map $f : W_1 \longrightarrow W_2$ with commutativity in the diagram

$$\begin{tikzcd} W_1 \arrow[rr, "f"] \arrow[dr, "\rho_1"'] & & W_2 \arrow[dl, "\rho_2"] \\ & X & \end{tikzcd} \tag{14.4}$$

f is not required to be proper. According to (13.5) a morphism $f : W_1 \longrightarrow W_2$ induces a map $f_! : K_i[C_0(\tau_1^*)\times\Gamma] \longrightarrow K_i[C_0(\tau_2^*)\times\Gamma]$.

(14.5) *Definition*. $K^i(X,\Gamma) = \underset{\mathscr{C}(X,\Gamma)}{\underrightarrow{\text{limit}}} K_i[C_0(\tau^*)\times\Gamma]$. In (14.5) the limit is taken using the $f_!$ maps of (13.5).

Let $F_i(X,\Gamma)$ be the free abelian group generated by all[1] triples (W,ρ,ξ) such that (W,ρ) is an object of $\mathscr{C}(X,\Gamma)$ and $\xi \in K_i[C_0(\tau^*)\times\Gamma]$. $R_i(X,\Gamma)$ denotes the subgroup of $F_i(X,\Gamma)$ generated by all elements in $F_i(X,\Gamma)$ of the form:

(i) $(W,\rho,\xi+\eta) - (W,\rho,\xi) - (W,\rho,\eta)$

(ii) $(W_1,\rho_1,\xi) - (W_2,\rho_2,f_!\xi)$

Then definition (14.5) is:

$$K^i(X,\Gamma) = F_i(X,\Gamma)/R_i(X,\Gamma) \tag{14.6}$$

If the action of Γ on X is proper, $(X,1_X)$ is a final object in $\mathscr{C}(X,\Gamma)$. 1_X is the identity map of X. It is then immediate from (14.5) and (14.6) that for a proper action $K^i(X,\Gamma) = K_i[C_0(X)\times\Gamma]$.

For an improper action, define $\mu : K^i(X,\Gamma) \longrightarrow K_i[C_0(X)\times\Gamma]$ by:

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[1] To avoid set-theoretic difficulties take W to be a C^∞ manifold (which is a closed subset) of some Euclidean space $\mathbb{R}^m$. This is possible by the Whitney embedding theorem.

$$\mu(W,\rho,\xi) = (\rho\pi)_!(\xi) \tag{14.7}$$

The commutativity of (13.11) implies that μ is well-defined.

(14.8) *Conjecture*. Let Γ act by diffeomorphisms on a C^∞ manifold X. Then $\mu : K^i(X,\Gamma) \longrightarrow K_i[C_0(X)\times\Gamma]$ is an isomorphism of abelian groups. $i = 0,1$.

Remark. (14.8) is part of a much more general conjecture [7,8]. For a C^* algebra A which may be:

(a) A reduced crossed-product twisted by a 2-cocycle.

(b) A reduced crossed-product arising from a C^∞ action of a Lie group on a manifold.

(c) The C^* algebra of a foliation.

We define a geometric K theory and conjecture that the geometric K theory is isomorphic to the K theory of the C^* algebra A.

Let $\mathscr{A}(X,\Gamma)$ be the full subcategory of $\mathscr{C}(X,\Gamma)$ whose objects are all (W,ρ) with the action of Γ on W free. For such (W,ρ), W is a principal Γ bundle over W/Γ. $K_i^{TX}(X\times_\Gamma E\Gamma)$ denotes the i-th K homology of $X \times_\Gamma E\Gamma$, twisted by $TX \times_\Gamma E\Gamma$. Using the (M,E,φ) definition of K homology given in [9], it is not difficult to prove

$$\underset{\substack{\longrightarrow \\ \mathscr{A}(X,\Gamma)}}{\text{limit}} K_i[C_0(\tau^*)\rtimes\Gamma] = K_i^{TX}(X\times_\Gamma E\Gamma) \tag{14.9}$$

The inclusion $\mathscr{A}(X,\Gamma) \subset \mathscr{C}(X,\Gamma)$ induces a map $K_i^{TX}(X\times_\Gamma E\Gamma) \longrightarrow K^i(X,\Gamma)$.

(14.10) *Lemma*. If all the isotropy groups for the action of Γ on X are torsion free, then $K_i^{TX}(X\times E\Gamma)_\Gamma \longrightarrow K^i(X,\Gamma)$ is an isomorphism.

Proof. In this case $\mathcal{A}(X,\Gamma) = \mathcal{C}(X,\Gamma)$.

Together Lemma (14.10) and Conjecture (14.8) become

(14.11) *Conjecture*. If all the isotropy groups for the action of Γ on X are torsion free, then $\mu : K_i^{TX}(X\times E\Gamma)_\Gamma \longrightarrow K_i[C_0(X)\rtimes\Gamma]$ is an isomorphism.

If Γ is torsion free and X is a point, $K^i(\cdot,\Gamma) = K_i(B\Gamma)$, where $B\Gamma$ is the classifying space of Γ. $C_0(\cdot) \rtimes \Gamma = C_r^*\Gamma$, the reduced C^* algebra of Γ. Due to [2], for Γ torsion free surjectivity of $\mu : K_0(B\Gamma) \longrightarrow K_0[C_r^*\Gamma]$ implies that there are no projections (other than 0 and 1) in $C_r^*\Gamma$.

Suppose that Γ (which may have torsion) acts on a tree T without inversion [32]. Let Γ act by a C^∞ action on X. Let $G \subset \Gamma$ be an isotropy group for an edge or a vertex of T. Assume that for all such G $\mu : K^i(X,G) \longrightarrow K_i[C_0(X)\times G]$ is an isomorphism. A remarkable recent result of M. Pimsner [28] then implies that μ : $K^i(X,\Gamma) \longrightarrow K_i[C_0(X)\rtimes\Gamma]$ is an isomorphism.

§15. Chern character for improper actions

$\mathrm{ch}_\Gamma : K^i(X,\Gamma) \longrightarrow H^i(X,\Gamma)$ is:

$$\mathrm{ch}_\Gamma(W,\rho,\xi) = (\rho\pi)_*(\mathrm{ch}_\Gamma(\xi) \cup \mathrm{Td}(\tau^*,\Gamma)) \tag{15.1}$$

Proposition (13.12) and the commutativity of (13.4) imply that ch_Γ : $K^i(X,\Gamma) \longrightarrow H^i(X,\Gamma)$ is well-defined.

(15.2) *Proposition.* For $i = 0,1$ ch_Γ gives an isomorphism of $\mathbb{C}$ vector spaces $K^i(X,\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow H^i(X,\Gamma)$.

Proof. Let (W,ρ) be an object of $\mathscr{C}(X,\Gamma)$. Map $K_i[C_0(\tau^*)\rtimes\Gamma]$ to $H^i(\tau^*,\Gamma)$ by

$$\xi \longmapsto \mathrm{ch}_\Gamma(\xi) \cup \mathrm{Td}(\tau^*,\Gamma) \tag{15.3}$$

Theorem (7.25) implies that (15.3) is an isomorphism $K_i[C_0(\tau^*)\times\Gamma] \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow H^i(\tau^*,\Gamma)$. According to (13.12) this passes to the limit to yield an isomorphism

$$\varinjlim_{\mathscr{C}(X,\Gamma)} (K_i[C_0(\tau^*)\rtimes\Gamma]\otimes_{\mathbb{Z}}\mathbb{C}) \longrightarrow \varinjlim_{\mathscr{C}(X,\Gamma)} H^i(\tau^*,\Gamma) \tag{15.4}$$

For the left side of (15.4) the limit is taken using the $f_!$ maps of (13.5). Thus the left side is $K^i(X,\Gamma) \otimes_{\mathbb{Z}} \mathbb{C}$. For the right side of (15.4) the limit is taken using the Gysin maps f_* of (13.3)

Commutativity of (13.4) implies that integration over the fiber gives a map

$$\varinjlim_{\mathscr{C}(X,\Gamma)} H^i(\tau^*,\Gamma) \longrightarrow H^i(X,\Gamma) \tag{15.5}$$

The proof is completed by showing that (15.5) is an isomorphism. This is done by using the classifying space for proper actions described in Appendix 3 below. Q.E.D.

We now define a map $\underline{\mu} : H^i(X,\Gamma) \longrightarrow K_i[C_0(X)\rtimes\Gamma] \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ by requiring commutativity in the diagram

$$\begin{array}{ccc} K^i(X,\Gamma) & \xrightarrow{\mu} & K_i[C_0(X)\rtimes\Gamma] \\ \Big\downarrow & & \Big\downarrow \\ H^i(X,\Gamma) & \xrightarrow[\underline{\mu}]{} & K_i[C_0(X)\rtimes\Gamma] \underset{\mathbb{Z}}{\otimes} \mathbb{C} \end{array} \tag{15.6}$$

In (15.6) the right vertical arrow is the tautological map.

(15.7) *Conjecture*. For $i = 0,1$ $\underline{\mu} : H^i(X,\Gamma) \longrightarrow K_i[C_0(X)\rtimes\Gamma] \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ is an isomorphism of vector spaces over $\mathbb{C}$.

Let $L \subset \Gamma$ be a subset of Γ such that:

(i) All $\gamma \in L$ are of finite order.

(ii) Any $\gamma \in \Gamma$ with γ of finite order is conjugate to one and only one element of L.

The identity element of Γ is in L. $Z(\gamma)$ denotes the centralizer of γ in Γ. $Z(\gamma)$ acts on X^γ. Then:

$$H^0(X,\Gamma) = \bigoplus_{\gamma\in L} \bigoplus_{j\in\mathbb{N}} H_{2j}^{TX^\gamma}(X^\gamma \times_{Z(\gamma)} EZ(\gamma))$$

$$H^1(X,\Gamma) = \bigoplus_{\gamma\in L} \bigoplus_{j\in\mathbb{N}} H_{2j+1}^{TX^\gamma}(X^\gamma \times_{Z(\gamma)} EZ(\gamma)) \tag{15.8}$$

In (15.8) $H_j^{TX^\gamma}(X^\gamma \times_{Z(\gamma)} EZ(\gamma))$ is the j-th homology of $X^\gamma \times_{Z(\gamma)} EZ(\gamma)$, twisted by $TX^\gamma \times_{Z(\gamma)} EZ(\gamma)$.

Remark. Consider the special case when all the isotropy groups for the action of Γ on X are torsion free. (15.8) simplifies to

$$H^0(X,\Gamma) = \bigoplus_{j\in\mathbb{N}} H_{2j}^{TX}(X\times_\Gamma E\Gamma)$$

$$H^1(X,\Gamma) = \bigoplus_{j\in\mathbb{N}} H_{2j+1}^{TX}(X\times_\Gamma E\Gamma). \tag{15.9}$$

According to (14.10) $K^i(X,\Gamma) = K_i^{TX}(X\times_\Gamma E\Gamma)$. Thus when all isotropy groups are torsion free $ch_\Gamma : K^i(X,\Gamma) \longrightarrow H^i(X,\Gamma)$ is just the ordinary homology Chern character (see §11 above):

$$ch : K_0^{TX}(X\times_\Gamma E\Gamma) \longrightarrow \bigoplus_{j\in\mathbb{N}} H_{2j}^{TX}(X\times_\Gamma E\Gamma)$$

$$ch : K_1^{TX}(X\times_\Gamma E\Gamma) \longrightarrow \bigoplus_{j\in\mathbb{N}} H_{2j+1}^{TX}(X\times_\Gamma E\Gamma)$$

Suppose that X is a point, and that Γ may have torsion. $S(\Gamma)$ is the set of all elements of finite order in Γ. The identity element of Γ is in $S(\Gamma)$. $F\Gamma$ denotes the Γ-module whose elements are all finite formal sums $\sum_{\gamma\in S(\gamma)} \lambda_\gamma[\gamma]$, with $\lambda_\gamma \in \mathbb{C}$. Γ acts on $F\Gamma$ by conjugation

$$\left[\sum_{\gamma\in S(\gamma)} \lambda_\gamma[\gamma]\right]\alpha = \sum_{\gamma\in S(\gamma)} \lambda_\gamma[\alpha^{-1}\gamma\alpha]$$

$H_j(\Gamma, F\Gamma)$ denotes the j-th homology group of Γ with coefficients $F\Gamma$. Set

$$H_{ev}(\Gamma, F\Gamma) = \bigoplus_{j\in\mathbb{N}} H_{2j}(\Gamma, F\Gamma)$$

$$H_{odd}(\Gamma, F\Gamma) = \bigoplus_{j\in\mathbb{N}} H_{2j+1}(\Gamma, F\Gamma)$$

When X is a point, Conjecture (15.7) becomes

(15.10) *Conjecture*. For any group Γ,

$$\underline{\mu} : H_{ev}(\Gamma, F\Gamma) \longrightarrow K_0[C_r^*\Gamma] \otimes_{\mathbb{Z}} \mathbb{C}$$

$$\underline{\mu} : H_{odd}(\Gamma, F\Gamma) \longrightarrow K_1[C_r^*\Gamma] \otimes_{\mathbb{Z}} \mathbb{C}$$

are isomorphisms of vector spaces over $\mathbb{C}$.

With $L \subset \Gamma$ as above, let $H_j(Z(\gamma);\mathbb{C})$ be the j-th homology group of $Z(\gamma)$ with trivial action on $\mathbb{C}$. Then

$$\begin{aligned} H_{ev}(\Gamma, F\Gamma) &= \bigoplus_{\gamma\in L} \bigoplus_{j\in\mathbb{N}} H_{2j}(Z(\gamma);\mathbb{C}) \\ H_{odd}(\Gamma, F\Gamma) &= \bigoplus_{\gamma\in L} \bigoplus_{j\in\mathbb{N}} H_{2j+1}(Z(\gamma);\mathbb{C}) \end{aligned} \tag{15.11}$$

(15.11) can be viewed as a special case of (15.8) or can be obtained directly as a straightforward application of Shapiro's lemma [14].

Appendix 1: $C_0(X) \times \Gamma$. Proof of Lemma (7.5)

Let S be a set. $\ell^2 S$ is the Hilbert space of all functions $\theta : S \longrightarrow \mathbb{C}$ such that:

(i) θ has countable support.

(ii) $\sum_{s \in S} |\theta(s)|^2 < \infty$

The inner product in ℓ^2 is:

$$\langle \theta, \xi \rangle = \sum_{s \in S} \theta(s)\overline{\xi(s)}$$

Let $u : S \times S \longrightarrow \mathbb{C}$ be a function. Fix $s \in S$. $i(s)u$ is the function from S to $\mathbb{C}$:

$$i(s)u : S \longrightarrow \mathbb{C}$$

defined by $i(s)u(t) = u(s,t)$. Suppose that $u : S \times S \longrightarrow \mathbb{C}$ satisfies:

For every $s \in S$, $i(s)u \in \ell^2 S$. Given such a u, let $\theta \in \ell^2 S$. $u*\theta$ is the function from S to $\mathbb{C}$ defined by $(U*\theta)(s) = \sum_{t \in S} u(s,t)\theta(t)$.

$\mathcal{M}S$ denotes the C^* algebra of all functions $u : S \times S \longrightarrow \mathbb{C}$ with:

(i) For every $s \in S$, $i(s)u \in \ell^2 S$.

(ii) For every $\theta \in \ell^2S$, $u * \theta \in \ell^2S$.

Addition, multiplication, norm, and $*$ in $\mathcal{M}S$ are:

$$(uv)(s,t) = \sum_{r \in S} u(s,r)v(r,t)$$

$$\|u\| = \sup_{\|\theta\|=1} \|u*\theta\|$$

$$u^*(s,t) = \overline{u(t,s)}$$

Each $u \in \mathcal{M}S$ determines a bounded operator T_u on ℓ^2S,

$$T_u\theta = u*\theta$$

$u \longmapsto T_u$ is an isomorphism of C^* algebras $\mathcal{M}S \longrightarrow \mathcal{L}(\ell^2S)$.

$\mathcal{M}_cS$ is the sub-algebra of $\mathcal{M}S$ whose elements are all $u : S \times S \longrightarrow \mathbb{C}$ with finite support. $\mathcal{M}_c^+S$ is the sub-algebra of $\mathcal{M}S$ generated by $\mathcal{M}_cS$ and the unit of $\mathcal{M}S$. $\mathcal{K}S$ is the sub-algebra of $\mathcal{M}S$ whose elements are all $u \in \mathcal{M}S$ with T_u a compact operator on ℓ^2S. $\mathcal{K}^+S$ is the sub-algebra of $\mathcal{M}S$ generated by $\mathcal{K}S$ and the unit of $\mathcal{M}S$. $\mathcal{K}S$ and $\mathcal{K}^+S$ are norm closed in $\mathcal{M}S$.

Lemma 1. $\mathcal{M}_c^+S$ is a dense sub-algebra of $\mathcal{K}^+S$ closed under holomorphic functional calculus.

Proof. For $s \in S$, let δ_s be:

$$\delta_s(t) = \begin{cases} 1 & t = s \\ 0 & t \neq s \end{cases}$$

$u \longmapsto T_u$ identifies $\mathcal{M}_c^+S$ with all bounded operators on ℓ^2S of the form $\lambda I + k$ where $\lambda \in \mathbb{C}$, I is the identity operator of ℓ^2S, and k is an operator such that $\langle k\delta_s, \delta_t\rangle$ is non-zero for only finitely many operator such that $\langle k\delta_s, \delta_t\rangle$ is non-zero for only finitely many $(s,t) \in S \times S$.

$u \longmapsto T_u$ identifies $\mathcal{K}^+S$ with all bounded operators on ℓ^2S of the form $\lambda I + k$ where $\lambda \in \mathbb{C}$, I is the identity operator of ℓ^2S, and k is a compact operator on ℓ^2S. The lemma is now evident.

Q.E.D.

Let $C_c^\infty(X,\Gamma)$ be the algebra of §7 above. An element of $C_c^\infty(X,\Gamma)$ is a finite formal sum $\sum_{\gamma\in\Gamma} f_\gamma[\gamma]$, where each $f_\gamma \in C_c^\infty(X)$. Note that the action of Γ on X is not assumed to be proper. Fix a point $x \in X$. Given $\left[\sum_{\gamma\in\Gamma} f_\gamma[\gamma]\right] \in C_c^\infty(X,\Gamma)$ let $\psi_x\left[\sum_{\gamma\in\Gamma} f_\gamma[\gamma]\right] \in \mathcal{M}\Gamma$ be:

$$\psi_x\left[\sum_{\gamma\in\Gamma} f_\gamma[\gamma]\right](\alpha,\beta) = f_{\alpha^{-1}\beta}(x\alpha)$$

$\psi_x : C_c^\infty(X,\Gamma) \longrightarrow \mathcal{M}\Gamma$ is an algebra homomorphism. Define $\|\ \|$ on $C_c^\infty(X,\Gamma)$ by

$$\|\eta\| = \sup_{x\in X} \|\psi_x\eta\| \qquad \eta \in C_c^\infty(X,\Gamma)$$

$C_0(X) \rtimes \Gamma$ is the completion of $C_c^\infty(X,\Gamma)$ in this norm. $C_0(X) \rtimes \Gamma$ is the reduced crossed-product C^* algebra arising from the action of Γ on $C_0(X)$. The $*$ in $C_0(X) \rtimes \Gamma$ is:

$$\left[\sum_{\gamma\in\Gamma} f_\gamma[\gamma]\right]^* = \sum_{\gamma\in\Gamma} \bar{f}^{\gamma}_{\gamma^{-1}}[\gamma]$$

ψ_x becomes a homomorphism of C^* algebras

$$\psi_x : C_0(X) \rtimes \Gamma \longrightarrow \mathcal{M}\Gamma$$

Thus each $\eta \in C_0(X) \rtimes \Gamma$ determines a continuous function φ_η from X to $\mathcal{M}\Gamma$.

$$\varphi_\eta(X) = \psi_x(\eta)$$

For $u \in \mathcal{M}\Gamma$ and $\gamma \in \Gamma$, let $u\gamma \in \mathcal{M}\Gamma$ be:

$$(u\gamma)(\alpha,\beta) = u(\gamma\alpha,\gamma\beta)$$

Then $u \longmapsto u\gamma$ is a right action (by unital C^* algebra automorphisms) of Γ on $\mathcal{M}\Gamma$. An immediate check shows that for each $\eta \in C_0(X) \rtimes \Gamma$ and each $\gamma \in \Gamma$,

$$\varphi_\eta(x\gamma) = \varphi_\eta(x)\gamma$$

Thus $\varphi_\eta : X \longrightarrow \mathcal{M}\Gamma$ is a continuous Γ-equivariant map. The action of Γ on $\mathcal{M}\Gamma$ preserves $\mathcal{M}_c\Gamma$, $\mathcal{M}_c^+\Gamma$, $\mathcal{K}\Gamma$, and $\mathcal{K}^+\Gamma$.

Assume now that the action of Γ on X is proper. $A(X,\mathcal{M}_c\Gamma)$ denotes the algebra of all C^∞ Γ-equivariant functions $\varphi : X \longrightarrow \mathcal{M}\Gamma$ with:

(i) $\varphi(x) \in \mathcal{M}_c\Gamma$ for all $x \in X$.

(ii) φ has Γ-compact support.

(iii) There exists a positive integer n (depending on φ) such that for all $x \in X$, Support $\varphi(x)$ has at most n elements.

$X^+ = X \cup \{+\}$ is the one-point compactification of X. Extend the action of Γ on X to X^+ by requiring that each $\gamma \in \Gamma$ fix the point at infinity.

$$+\gamma = +$$

Let $A(X,\mathcal{K}\Gamma)$ be the C^* algebra of all continuous Γ-equivariant functions $\varphi : X^+ \longrightarrow \mathcal{K}\Gamma$ with $\varphi(+) = 0$.

Lemma 2. If the action of Γ on X is proper, then $\eta \longmapsto \varphi_n$ gives algebra isomorphisms

$$C_c^\infty(X,\Gamma) \cong A(X,\mathcal{M}_c\Gamma)$$

$$C_0(X) \rtimes \Gamma \cong A(X,\mathcal{K}\Gamma)$$

Proof. $\eta \longmapsto \varphi_\eta$ injects $C_c^\infty(X,\Gamma)$ into $A(X,\mathcal{M}_c\Gamma)$. For the surjectivity, suppose $\varphi \in A(X,\mathcal{M}_c\Gamma)$. Define $\eta = \Sigma f_\gamma[\gamma]$ by

$$f_\gamma(x\alpha) = \varphi(x)(\alpha,\alpha\gamma)$$

Then $\eta \in C_c^\infty(X,\Gamma)$ and $\varphi_\eta = \varphi$.

It now follows that $\eta \longmapsto \varphi_\eta$ maps $C_0(X) \rtimes \Gamma$ into $A(X,\mathcal{K}\Gamma)$. Since $A(X,\mathcal{M}_c\Gamma)$ is dense in $A(X,\mathcal{K}\Gamma)$ this is an isomorphism of C^* algebras $C_0(X) \rtimes \Gamma \cong A(X,\mathcal{K}\Gamma)$. Q.E.D.

Δ denotes the unit of $\mathcal{M}\Gamma$.

$$\Delta(\alpha,\beta) = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$$

Let $A(X,\mathcal{K}^{+}\Gamma)$ be the C^{*} algebra of all continuous Γ-equivariant functions $\varphi : X^{+} \longrightarrow \mathcal{K}^{+}\Gamma$ such that there exists $\lambda \in \mathbb{C}$ with $\varphi(+) = \lambda\Delta$ and $\varphi(x) - \lambda\Delta \in \mathcal{K}\Gamma$ for all $x \in X$.

Similarly let $A(X,\mathcal{M}_{c}^{+}\Gamma)$ be the algebra of all C^{∞} Γ-equivariant functions $\varphi : X \longrightarrow \mathcal{M}_{c}^{+}\Gamma$ such that there exists $\lambda \in \mathbb{C}$ with $x \longmapsto \varphi(x) - \lambda\Delta$ an element of $A(X,\mathcal{M}_{c}\Gamma)$.

$C_{c}^{\infty}(X,\Gamma)^{+}$ and $C_{0}(X) \times \Gamma^{+}$ are $C_{c}^{\infty}(X,\Gamma)$ and $C_{0}(X) \times \Gamma$ with unit adjoined.

Lemma 3. $C_{c}^{\infty}(X,\Gamma)^{+}$ is a dense sub-algebra of $C_{0}(X) \rtimes \Gamma^{+}$ closed under holomorphic functional calculus.

Proof. According to lemma 2 $\eta \longmapsto \varphi_{\eta}$ gives isomorphisms

$$C_{c}^{\infty}(X,\Gamma)^{+} \cong A(X,\mathcal{M}_{c}^{+}\Gamma)$$

$$C_{0}(X) \rtimes \Gamma^{+} \cong A(X,\mathcal{K}^{+}\Gamma)$$

In $A(X,\mathcal{K}^{+}\Gamma)$ holomorphic functional calculus is done pointwise. The proof is now completed by using lemma 1. Q.E.D.

Lemma (7.5) has been proved.

Appendix 2: Γ uncountable

Some minor technical changes are needed in order to state conjecture (14.8) for an uncountable discrete group Γ. Let Γ be such a group. Assume that Γ acts by diffeomorphisms on a C^∞ manifold X.

$$X \times \Gamma \longrightarrow X$$

There is a slight difficulty in defining $K^i(X,\Gamma)$ because Γ cannot act properly on a C^∞ manifold W which satisfies the second axiom of countability (i.e. there is a countable collection of open sets of W such that any open set is the union of sets in this countable collection). Hence in defining $K^i(X,\Gamma)$ we must allow C^∞ manifolds W which are not second countable. However, we shall require that the orbifold W/Γ is second countable.

Let $\mathscr{C}(X,\Gamma)$ be the category of all pairs (W,ρ) such that:

(i) W is a C^∞ manifold with a given proper C^∞ action of Γ.

(ii) $\rho : W \longrightarrow X$ is a C^∞ Γ-equivariant submersion mapping W onto X.

(iii) W/Γ is second countable.

Then $K^i(X,\Gamma)$ is defined

$$K^i(X,\Gamma) \; \underset{\mathscr{C}(X,\Gamma)}{\underrightarrow{\text{limit}}} \; K_i[C_0(\tau^*)\rtimes\Gamma]$$

$H^i(X,\Gamma)$, $K_i[C_0(X)\rtimes\Gamma]$, and $ch_\Gamma : K^i(X,\Gamma) \longrightarrow H^i(X,\Gamma)$ are as in the countable case. There is then the commutative diagram

$$\begin{array}{ccc} K^i(X,\Gamma) & \xrightarrow{\;\mu\;} & K_i[C_0(X)\rtimes\Gamma] \\ \Big\downarrow{\scriptstyle ch_\Gamma} & & \Big\downarrow \\ H^i(X,\Gamma) & \xrightarrow[\;\underline{\mu}\;]{} & K_i[C_0(X)\rtimes\Gamma] \underset{\mathbb{Z}}{\otimes} \mathbb{C} \end{array}$$

and we conjecture that μ and $\underline{\mu}$ are isomorphisms.

Equivalently, let $\mathcal{H}(\Gamma)$ be the category whose objects are all countable subgroups of Γ. A morphism in $\mathcal{H}(\Gamma)$ is an inclusion $H_1 \subset H_2$. Each such inclusion produces a commutative diagram

$$\begin{tikzcd}
K^i(X,H_1) \arrow[r, "\mu"] \arrow[d] & K_i[C_0(X)\rtimes H_1] \arrow[d] \\
K^i(X,H_2) \arrow[r, "\mu"'] & K_i[C_0(X)\rtimes H_2]
\end{tikzcd}$$

and we then have

$$K^i(X,\Gamma) = \underrightarrow{\underline{\mathrm{limit}}}_{\mathcal{H}(\Gamma)} \; K^i(X,H)$$

$$K_i[C_0(X)\rtimes\Gamma] = \underrightarrow{\underline{\mathrm{limit}}}_{\mathcal{H}(\Gamma)} \; K_i[C_0(X)\rtimes H]$$

The case of $U(n)_\delta$ acting on $U(n)$ is relevant to the study of higher η-invariants. Here $U(n)$ is the Lie group of all $n \times n$ unitary matrices of complex numbers. $U(n)_\delta$ is $U(n)$ with the discrete topology. $U(n)_\delta$ acts on $U(n)$ by right multiplication.

$$U(n) \times U(n)_\delta \longrightarrow U(n)$$

Appendix 3: Classifying space for proper actions

Y is a CW complex and Γ is a discrete group. A proper Γ-space over Y is a topological space Z with a given proper action of Γ and a map $\pi : Z \longrightarrow Y$ such that:

(i) $\pi(z\gamma) = \pi z$ for all $(z,\gamma) \in Z \times \Gamma$

(ii) The map $Z/\Gamma \longrightarrow Y$ determined by π is a homeomorphism of Z/Γ onto Y.

Two proper Γ-spaces (Z,π) and (Z',π') over Y are *isomorphic* if there exists a Γ-equivariant homeomorphism $h : Z \longrightarrow Z'$ with $\pi = \pi' h$.

Let $f : Y \longrightarrow V$ be a continuous map. Suppose that (Z,π) is a proper Γ-space over V. Consider the usual Cartesian square

$$\begin{array}{ccc} Y \times_V Z & \longrightarrow & Z \\ \Big\downarrow{\scriptstyle \tilde{\pi}} & & \Big\downarrow{\scriptstyle \pi} \\ Y & \xrightarrow[f]{} & V \end{array}$$

One checks easily that $(Y \times_V Z, \tilde{\pi})$ is a proper Γ-space over Y. Set $f^*(Z,\pi) = (Y \times_V Z, \tilde{\pi})$.

In particular, let (Z,π) be a proper Γ-space over $Y \times [0,1]$. Fix $t \in [0,1]$ and map Y into $Y \times [0,1]$ by $i_t : Y \longrightarrow Y \times [0,1]$

$$i_t(y) = (y,t)$$

$i_t^*(Z,\pi)$ is then a proper Γ-space over Y.

Definition. Two proper Γ-spaces (Z_0, π_0) and (Z_1, π_1) over Y are *homotopic* if there exists a proper Γ-space Z over $Y \times [0,1]$ with $i_0^*(Z,\pi)$ isomorphic to (Z_0, π_0) and $i_1^*(Z,\pi)$ isomorphic to (Z_1, π_1).

Let $P\Gamma(Y)$ be the set of homotopy classes of proper Γ-spaces over Y. $P\Gamma$ satisfies the axioms of [13]. Hence there is a universal example. That is, there is a space $\underline{E}\Gamma$ on which Γ acts properly such that setting $\underline{B}\Gamma = \underline{E}\Gamma/\Gamma$ we have:

(i) If (Z,π) is any proper Γ-space over Y, then there exists a continuous Γ-equivariant map $h : Z \longrightarrow \underline{E}\Gamma$

(ii) $P\Gamma(Y) = [Y, \underline{B}\Gamma]$ where $[Y, \underline{B}\Gamma]$ is the set of homotopy classes of continuous maps from Y to $\underline{B}\Gamma$.

Examples (1) If Γ is torsion free, any proper Γ action is principal so $\underline{E}\Gamma = E\Gamma$ and $\underline{B}\Gamma = B\Gamma$.

(2) If Γ is finite, any Γ action is proper and $\underline{E}\Gamma$, $\underline{B}\Gamma$ are one-point spaces. $\bullet = \underline{E}\Gamma = \underline{B}\Gamma$.

(3) Following [32] assume that Γ acts without inversion on a tree T. Assume that the isotropy group of each vertex is finite. Then $\underline{E}\Gamma = T$ and $\underline{B}\Gamma = T/\Gamma$.

(4) Let Γ be a discrete subgroup of a Lie group G. Assume that $\pi_0 G$ is finite and that $H\backslash G$ admits a G-invariant Riemannian metric with all sectional curvatures non-positive. Here H is the maximal compact subgroup of G. Then $\underline{E}\Gamma = H\backslash G$ and $\underline{B}\Gamma = H\backslash G/\Gamma$.

(5) Quite generally $\underline{E}\Gamma$ and $\underline{B}\Gamma$ can be constructed by the iterated join of [25]. Let $H_1, H_2, \ldots$ be finite subgroups of Γ such that any finite subgroup of Γ is contained in a conjugate of some H_i. (One could take $H_1, H_2, \ldots$ to be all the finite subgroups of Γ.)

Let Σ be the disjoint union $\Sigma = H_1\backslash\Gamma \cup H_2\backslash\Gamma \cup \dots$. Then $\underline{E}\Gamma$ is the infinite join of Σ:

$$\underline{E}\Gamma = \Sigma \circ \Sigma \circ \Sigma \circ \dots$$

$$\underline{B}\Gamma = (\Sigma \circ \Sigma \circ \Sigma \circ \dots)/\Gamma$$

$\underline{E}\Gamma$ and $\underline{B}\Gamma$ are used in proving that the map $\varinjlim_{\mathscr{C}(X,\Gamma)}$ $H^i(\tau^*,\Gamma) \longrightarrow H^i(X,\Gamma)$ of (15.5) is an isomorphism.

This isomorphism is established in three steps:

(1) $\displaystyle \varinjlim_{\mathscr{C}(X,\Gamma)} H^i(\tau^*,\Gamma) = \varinjlim_{\mathscr{C}(X,\Gamma)} H_i^{TX}(W,\Gamma)$

(2) $\displaystyle \varinjlim_{\mathscr{C}(X,\Gamma)} H_i^{TX}(W,\Gamma) = H_i^{TX}(X\times\underline{E}\Gamma,\Gamma)$

(3) $H_i^{TX}(X\times\underline{E}\Gamma,\Gamma) = H^i(X,\Gamma)$

The main point is that $X \times \underline{E}\Gamma$ behaves as if it were a final object for the category $\mathscr{C}(X,\Gamma)$. If (W,ρ) is an object in $\mathscr{C}(X,\Gamma)$, then W maps to $X \times \underline{E}\Gamma$ by a continuous Γ-equivariant map.

Note. If Γ acts properly on a topological space Y, then by definition:

$$H_0(Y,\Gamma) = \bigoplus_{j\in\mathbb{N}} H_{2j}(\hat{Y}/\Gamma;\mathbb{C})$$

$$H_1(Y,\Gamma) = \bigoplus_{j\in\mathbb{N}} H_{2j+1}(\hat{Y}/\Gamma;\mathbb{C})$$

If Γ acts properly on Y and E is a Γ-equivariant $\mathbb{R}$ vector bundle on Y, then by definition:

$$H_0^E(Y,\Gamma) = \bigoplus_{j\in\mathbb{N}} H_{2j}((\hat{E}/\Gamma,\hat{E-\{0\}}/\Gamma);\mathbb{C})$$

$$H_1^E(Y,\Gamma) = \bigoplus_{j\in\mathbb{N}} H_{2j+1}((\hat{E}/\Gamma,\hat{E-\{0\}}/\Gamma);\mathbb{C})$$

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