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EIGENVALUE ASYMPTOTICS FOR THE STURM-LIOUVILLE OPERATOR WITH POTENTIAL HAVING A STRONG LOCAL NEGATIVE SINGULARITY

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Abstract. We find asymptotic formulas for the eigenvalues of the Sturm-Liouville operator on the finite interval, with potential having a strong negative singularity at one endpoint. This is the case of limit circle in H. Weyl sense. We establish that, unlike the case of an infinite interval, the asymptotics for positive eigenvalues does not depend on the potential and it is the same as in the regular case. The asymptotics of the negative eigenvalues may depend on the potential quite strongly, however there are always asymptotically fewer negative eigenvalues than positive ones. By unknown reasons this type of problems had not been studied previously.

Keywords: Sturm-Liouville operator, singular potential, asymptotics of eigenvalues.

Mathematics Subject Classification: 34L20, 34L40.

1. INTRODUCTION

The study of spectral properties of the Sturm-Liouville operator has been attracting attention of researchers for more than a century. This study induced creation of various advanced methods of analysis, in particular of functional analysis. Sturm-Liouville spectral problems are related to some important applications in Physics. One of the best studied topics in the spectral theory of the Sturm-Liouville operators is the eigenvalue asymptotics. There are numerous publications in this field; we mention only the books [10–12, 14, 19].

Sturm-Liouville spectral problems are naturally divided into two classes. The problem

$$Hy \equiv -y'' + q(x)y = \lambda y, x \in I = (x_0, x_1), \tag{1.1}$$

with certain boundary conditions at the endpoints x_0, x_1 was initially called regular if the interval I is finite and the "potential" q is continuous on $\bar{I}$, otherwise the problem used to be called singular. In the regular case, the spectral theory can be usually reduced to some problems in complex analysis and algebra. The singular problems are much more complicated and they require some hard analysis.

The reasons for a problem to be singular may be the infiniteness of the interval I, the singularity of the potential q, or both. Usually it is supposed that the singularities of q are placed at the endpoints of the interval (including the infinity), and that q is nice in some proper sense at all interior points. Under these assumptions, the spectral theory on the qualitative level is fairly well understood. The important starting point is the criterium for whether some boundary conditions should be imposed at the endpoints. This question has an explicit answer in the terms of the "limit point – limit circle" classification, more exactly, in the terms of the behavior of solutions of the equation when approaching the endpoints, and this leads to rather sharp conditions in the terms of the properties of the potential q. As for the quantitative results, the state of the art is the following.

If $q(x)$ has no local singularities and tends to $+\infty$ at the infinity, the spectrum is discrete, consists of the sequence of eigenvalues tending to $+\infty$, and, under rather mild regularity conditions, has a regular asymptotic behavior. Namely, if $N(H;(\lambda_1,\lambda_2))$ denotes the number of eigenvalues of (1.1) in the interval (λ_1,λ_2), then

$$N(H;(0,\lambda)) \sim \pi^{-1} \int_{x_0}^{x_1} (\lambda - q(x))_+^{\frac{1}{2}} dx, \quad \lambda \to +\infty, \tag{1.2}$$

where $(\lambda - q(x))_+ = \max(\lambda - q(x), 0)$. Formula (1.2) is usually called the Weyl type one, or quasi-classical. The latter name expresses the fundamental relation between classical and quantum description of processes. In our case, one associates with the (quantum) operator H in (1.1) the corresponding classical Hamiltonian in $\mathbb{R}^2$: $\mathcal{H}(x,\xi) = \xi^2 + q(x)$. In this case, the right-hand side in (1.2) is exactly $(2\pi)^{-1}$ times the 2-dimensional volume in $\mathbb{R}^2$ of the region $\Omega(\lambda) = \{(x,\xi) \in \mathbb{R}^2 : \mathcal{H}(x,\xi) < \lambda\}$. This relation spreads to a very general setting, to the asymptotic formulas for the eigenvalues of elliptic (and not quite elliptic) differential and pseudodifferential operators in many dimensions, with extensions to operators on manifolds, bundles and even further.

Returning to the quasi-classical formula (1.2), we note that it is proved (and valid) under certain regularity conditions imposed on the potential $q(x)$, both forbidding very fast oscillations at interior points of the interval and setting some restrictions on the behavior at infinity. The typical local regularity condition here is

$$q'(x) = O(q(x)^d), \quad x \to \infty, d < 3/2, \tag{1.3}$$

Additionally, some "global" conditions should be imposed, requiring a certain regularity in the behavior of the right-hand side of (1.2) as a function of λ. If the

local or global conditions are broken, the formula (1.2) may fail. However, the local irregularities can be sometimes taken care of, by means of replacing $q(x)$ in the expression on the right-hand side of (1.2) by its regularized version q^* - see Ch. VII in [10, 13] or later papers by the authors of [13] (see also [16], where the differentiability condition is considerably relaxed.)

Another case, where the asymptotics of eigenvalues is rather well understood, is the case of a locally regular potential $q(x)$ that tends to zero at infinity. In this case, the positive spectrum of the operator H is continuous and coincides with the positive half-line, while the negative spectrum consists of finitely or infinitely many eigenvalues (there may be no eigenvalues at all). If there are infinitely many eigenvalues (this usually means that the potential is essentially negative and tends to zero at infinity not too rapidly), a formula, similar to (1.2), holds for $N(H; (\lambda, -\infty))$, with the parameter λ tending to -0. Again, certain local and global regularity conditions should be imposed, and the eigenvalue formula is semi-classical, immediately, or with replacement of the potential q by its regularized version.

In both cases above, the need for setting boundary conditions at the endpoints of the interval I is determined rather easily. Say, for the (typical) case of I being a semi-axis $(0, \infty)$ and the potential being bounded at infinity, to define the self-adjoint operator, one does not need to set a boundary condition at infinity, and at the finite endpoint this issue is determined by the behavior of $q(x)$ as $x \to 0$. Under some minor regularity conditions, if $q(x) \to +\infty$ faster than x^{-2} as $x \to +0$, the point zero is of limit point type, so no boundary condition at 0 is needed. On the other hand, if $q(x)$ is bounded near 0, or grows slower than x^{-2}, one needs to set a boundary condition at zero- it is of limit circle type. The infinite endpoint is always of limit point type, as long as the potential is lower semibounded.

Before we pass to the case of the potential being not lower semibounded, we refer to an interesting discussion of the above topic in the classical book [15], Section X.1A. Here the property of the "limit point" is called "quantum completeness", which corresponds to the self-adjointness of the operator. The meaning of self-adjointness consists in the fact that in the quantum complete case, the evolutional Schrödinger equation $-i\partial u/\partial t = Hu$ (with proper initial condition) has a unique solution, global in t. This property is compared with the "classical completeness": the existence of the solution, global in t of the equation $x''(t) = -dq/dx$ for any initial data $x(0), x'(0)$ with $x(0) \in I$. This means that a classical "particle", starting its movement in the field with potential $q(x)$ at a point $x(0)$ with some initial velocity, never, in a finite time, reaches an endpoint of the interval nor goes to infinity. Thus, the motion of the "particle" is defined for all times. On the other hand, if the motion is not classically complete, i.e., the particle at $t = t_c$ reaches an endpoint or goes to infinity, in order to define its motion after $t = t_c$, we need to describe a "law of reflection", which means that the particle reflects from the endpoint or infinity, but, possibly, with the phase change. The choice of this law corresponds to the boundary condition one has to set at the boundary point of limit circle type, and the limit circle itself parameterizes these reflection laws. The discussion of this issue in [15] demonstrates, however, certain limitations in the treatment of two types of completeness.

Now we pass to the case when the potential $q(x)$ is not (essentially) lower semi-bounded. We denote $h(x) = -q(x)$ and suppose that $h(x) \to +\infty$ as x approaches the endpoint ∞ of the interval $I = (x_0, \infty)$. Suppose that $h'(x)$ has constant sign at infinity. From the point of view of the classical completeness, the particle is accelerated by the potential $q(x)$, as it tends to infinity. So, the classical completeness question is resolved by finding out, if it accelerates sufficiently fast to reach the endpoint in a finite time – or not. It was found out in [17] that, again under some local and global regularity assumptions, the required rate of growth of $h(x), x \to \infty$, is determined by the integral $\int^{\infty}(K + h(x))^{-\frac{1}{2}}dx$ for K sufficiently large: if this integral diverges, then $x = \infty$ is of limit point type, and it is of limit-circle type otherwise, i.e., if this integral converges. Moreover, the spectral properties in these two cases differ drastically: in the former case, the spectrum covers the whole real axis, while in the latter case the spectrum is discrete and not semibounded. Here, again, the conditions on the potential granting quantum completeness coincide with the ones for the classical completeness, provided sufficient local regularity is supposed. Note that here, in the limit-circle case, the above "phase change" picture does not work directly, since the solutions oscillate rapidly when approaching the singular endpoint. So, the additionally boundary condition is set by choosing those functions as belonging to the domain of the operator, for which the solutions of the equation, with these functions on the right-hand side, are 'almost' proportional to a chosen solution of the homogeneous equation - this latter solution parameterizes self-adjoint boundary conditions.

The study of the asymptotic behavior of eigenvalues of the Sturm-Liouville operator on $(0, \infty)$, with $q(x)$ bounded near zero and tending to $-\infty$ at infinity sufficiently fast, so that the limit circle at infinity takes place, started in 1954, see [9]. The specifics of the problem required a new approach. The operator is not semi-bounded, so the variational method, very efficient for semi-bounded operators, could not be applied. Bookkeeping zeroes of solutions, also widely used for semi-bounded problems (the number of the eigenfunction for a regular problem is closely related to the quantity of its zeroes), could not be applied either, since all solutions oscillate rapidly at infinity and have infinitely many zeroes. P. Heywood found in [9] a modification of the zero-counting method. First, the problem on the finite interval $(0, b)$ was considered, with some boundary conditions set at the point b. The corresponding operator is denoted H_b. For fixed $\lambda > 0, -\mu < 0$, the quantities $N(H_b; (0, \lambda))$ and $N(H_b; (-\mu, 0))$ are studied. This is achieved by evaluating the number of zeroes $n(b, s)$ of the solutions of the equation $(H - s)y = 0$ on the interval $(0, b)$ for $s = 0, s = \lambda$ and $s = -\mu$. Although, as $b \to \infty$, each of these quantities grows unboundedly, the differences $n(b, \lambda) - n(b, 0)$ and $n(b, 0) - n(b, -\mu)$ turn out to be bounded, uniformly in b, and, moreover, they admit an explicit expression, not depending on b, with an error term, uniformly bounded in b. This information on the zeroes produces the expressions for $N(H_b; (0, \lambda))$ and $N(H_b; (-\mu, 0))$. The final step consists in proving that these expressions converge, as $b \to \infty$, to the corresponding counting functions for the operator on the semiaxis.

This result in [9] differs drastically from the ones for the semibounded case. In the 60 years that passed since, no semi-classical explanation of this formula was found. Some regularity conditions, to be specified later on, are imposed, the first one being the (eventual) monotonicity of the potential $q(x)$, and the asymptotic formulas, with

$h(p(\mu)) = \mu$, are

$$N(H,(0,\lambda)) = \pi^{-1}\int_0^\infty [(\lambda + h(x))^{\frac{1}{2}} - h(x)^{\frac{1}{2}}]dx + O(1),\ \lambda > 0, \tag{1.4}$$

$$N(H,(-\mu,0)) = \pi^{-1}\int_0^{p(\mu)} h(x)^{\frac{1}{2}}dx + \pi^{-1}\int_{p(\mu)}^\infty [h(x)^{\frac{1}{2}} - (h(x)-\mu)^{\frac{1}{2}}]dx + O(1). \tag{1.5}$$

Much later, in 1974, without (initially) knowing about [9], the problem of the eigenvalue asymptotics for the case $q(x) \to -\infty$ was considered by Belogrud and Kostuchenko, [7]. Actually, a short note, without proofs, appeared in [7], but a more detailed exposition was published in (now inaccessible) [6], with the final presentation filling chapters 5 and 9 in the book [10]. An approach, different from the one in [9] was used. An asymptotic expression, uniform both in λ and x, was found for the solutions of the equation $(H-\lambda)y = 0$ for λ in a special complex region. Using this result, an asymptotic formula was derived for the trace of the resolvent of H in this complex region. Finally, a specially constructed Tauberian theorem produced formulas (1.4), (1.5). The reasoning and calculations in [10] are much more complicated and laborious than the ones in [9], however the conditions imposed on the potential are less restrictive: no assumptions on the second derivative are made. Moreover, as it is shown in [10], the method used can be applied to some higher order non-semibounded operators.

Further activities in this topic concentrated in improving the asymptotic estimates. This is impossible to do in the terms of the counting function: since $N(H,(\lambda_1,\lambda_2))$ is an integer, a remainder estimate better than $O(1)$ is impossible. On the other hand, if a formula is found, expressing the eigenvalues themselves in an implicit form as solutions of some equations, such results can give improved asymptotic formulas for the eigenvalues, with a higher order of accuracy. The first result of this kind was obtained by Alenitsyn in [1]. By finding an asymptotic expression, with several terms, of solutions of the equation, using the WKB method, Alenitsyn derived two-term equations (for the positive and for the negative spectrum) determining the eigenvalues in an implicit form. An important feature of this sharpening is that one can trace the dependence of the eigenvalue asymptotics on the parameter fixing the self-adjoint extension by setting the boundary conditions at infinity - which was impossible by the previously used methods.

Some years later, a series of papers by F. Atkinson and C. Fulton appeared, see [2–4]. In the seminal paper [2] a new approach to non-semibounded problems is presented, based upon a modified Prüfer transform, reducing the second order linear equation to a system of first order nonlinear equations, for which the asymptotic analysis becomes more feasible. Besides deriving an improved Heywood formula, in Alenitsyn style, and demonstrating a number of interesting examples and consequences, the authors in [2] announce subsequent papers, [3–5], where the approach would be developed further,

in order to give algorithmically arbitrary many higher order terms in the implicit expression for the eigenvalues. The three cases announced are:

1. The problem on $(0,\infty)$ with $q(x)$ tending to $-\infty$ faster than $-x^2$ at infinity;
2. The problem on $(0,1)$ with $q(x)$ behaving like $Cx^{-\alpha}$, $\alpha \in [1,2)$ near zero;
3. The problem on $(0,1)$ with $q(x)$ behaving like $-x^{-\alpha}$, $\alpha > 2$ near zero.

The papers [3, 4], containing the analysis of the cases (1) and (2), have appeared. However, the paper [5], although announced several times, was never published.

Thus, a strange situation had arisen. A rather complete spectral analysis of the singular non-semibounded Sturm-Liouville operator on the semi-axis with singularity at infinity was performed long ago, while for the complementing case, the potential tending rapidly to $-\infty$ at the finite endpoint remains completely unresolved.

The present paper is devoted to filling this gap. We modify the approach initiated by Heywood and find the asymptotic formulas for the eigenvalue counting functions. We have in mind to investigate also the possibility of modifying the approach by Alenitsyn for the case in question, in order to obtain improved asymptotic eigenvalue formulas.

We need, however, to start by presenting (in Sect. 2) the version of Weyl's limit-point – limit-circle theory for a finite singular point. By some tradition, set quite long ago, in the exposition of this theory, one considers the infinite singular point, just mentioning that the case of a finite singular point is treated in a similar way. We check that, in fact, the reasoning is mainly similar (and we follow essentially the presentation in [19]), however some important details need to be elaborated anew. Then, in Sect. 3 we find sufficient analytical conditions on the singular negative potential q, granting that the finite endpoint is of limit-circle type. The main part of the paper is devoted to finding the asymptotic formulas for eigenvalues. These formulas show, in particular, the presence of a new effect, not existing for the limit-circle problem at infinity, considered previously. Namely, it turns out that the asymptotics for the positive eigenvalues, according to the formula obtained, is the same, at least in the leading term, for all potentials subject to the regularity conditions, in particular, the same as for the regular problem. On the other hand, the asymptotics for the negative eigenvalues depends essentially on the potential. Moreover, it turns out that, asymptotically, there are fewer negative eigenvalues than positive ones, on intervals of the same length. In the concluding part of the paper we present calculations for several interesting examples of potentials.

2. GENERAL CONSIDERATIONS. WEYL'S FUNCTION

In this section we present some, mostly known, formal relations concerning the Weyl function for Sturm-Liouville operator $H = -\frac{d^2}{dx^2} + q(x)$. These relations were obtained in [19] for semi-axis case. Here we repeat them for the case of a finite interval with a singularity at one end point. As it mentioned in [19], it was effectuated similarly.

From now on, we consider Sturm-Liouville spectral problem (1.1) with $x_0 = 0$, $x_1 = 1$, and a negative singularity at zero. Let $q(x)$ be a continuous real-valued

function on $I = (0,1]$, $(a,b] \subset I$. For complex numbers λ, λ' let $F(x)$, $G(x)$ be functions such that

$$(\lambda - H)F(x) = (\lambda' - H)G(x) = 0.$$

Then we have

$$\begin{aligned}(\lambda' - \lambda)\int_a^b F(x)G(x)dx &= \int_a^b [F(x)(q(x)G(x) - G''(x)) - G(x)(q(x)F(x) - F''(x))]dx \\ &= -\int_a^b [F(x)G''(x) - G(x)F''(x)]dx = W_a(F,G) - W_b(F,G),\end{aligned} \tag{2.1}$$

where

$$W_x(F,G) = \begin{vmatrix} F(x) & G(x) \\ F'(x) & G'(x) \end{vmatrix} \tag{2.2}$$

is the Wronskian of the functions F, G. If $\lambda' = \overline{\lambda}$, $b = 1$ and $G = \overline{F}$, this gives

$$2\operatorname{Im}\lambda \int_a^1 |F(x)|^2 dx = iW_a(F,\overline{F}) - iW_1(F,\overline{F}). \tag{2.3}$$

We consider two special solutions of (1.1). Let $\phi(x) = \phi(x,\lambda)$, $\theta(x) = \theta(x,\lambda)$ be the solutions of (1.1) satisfying the boundary conditions at the point $x = 1$:

$$\begin{aligned} \phi(1) &= \sin\alpha, & \phi'(1) &= -\cos\alpha, \\ \theta(1) &= \cos\alpha, & \theta'(1) &= \sin\alpha, \end{aligned}$$

where α is real. Then $W_x(\phi,\theta) = W_1(\phi,\theta) = 1$.

Using these solutions, we will obtain a number of relations connecting the (spectral) parameter λ and the parameter in the expression of the general solution via the solutions fixed above, when a certain boundary condition is set at a point $a \in I$.

The general solution, up to a constant factor, can be written as $u(x) = \theta(x) + l\phi(x)$. This solution satisfies the boundary condition

$$u(1)(\sin\alpha - l\cos\alpha) - u'(1)(\cos\alpha + l\sin\alpha) = 0 \tag{2.4}$$

at the right endpoint of our interval I.

Now we consider such solution that additionally satisfies the boundary condition at some point $x = a$, $a \in I$:

$$u(a)\cos\beta + u'(a)\sin\beta = (\theta(a) + l\phi(a))\cos\beta + (\theta'(a) + l\phi'(a))\sin\beta = 0, \tag{2.5}$$

where β is real. Of course, by varying β, we obtain all possible solutions of the equation, satisfying (2.4). On the other hand, having fixed some β in (2.5), we can find the parameter l in (2.4) by

$$l = l(\lambda) = -\frac{\theta(a)\cot\beta + \theta'(a)}{\phi(a)\cot\beta + \phi'(a)}. \tag{2.6}$$

For each a, as $\cot\beta$ varies, l draws a circle C_a in the complex plane. Replacing $\cot\beta$ in (2.6) by a complex variable z, we obtain

$$l = l(\lambda, z) = -\frac{\theta(a)z + \theta'(a)}{\phi(a)z + \phi'(a)}.$$

Here $l = \infty$ corresponds to $z = -\frac{\phi'(a)}{\phi(a)}$. Hence the center of C_a corresponds to the value $z = \overline{\left(-\frac{\phi'(a)}{\phi(a)}\right)}$, and therefore, at the center,

$$l = l\left(\lambda, -\frac{\overline{\phi'(a)}}{\overline{\phi(a)}}\right) = -\frac{-\theta(a)\overline{\phi'(a)} + \theta'(a)\overline{\phi(a)}}{-\phi(a)\overline{\phi'(a)} + \phi'(a)\overline{\phi(a)}} = -\frac{W_a(\theta,\overline{\phi})}{W_a(\phi,\overline{\phi})}.$$

Also, we have

$$\operatorname{Im}\left(-\frac{\phi'(a)}{\phi(a)}\right) = \frac{i}{2}\left(\frac{\phi'(a)}{\phi(a)} - \overline{\left(\frac{\phi'(a)}{\phi(a)}\right)}\right) = -\frac{i}{2}\frac{W_a(\phi,\overline{\phi})}{|\phi(a)|^2},$$

which has the same sign as $-\operatorname{Im}\lambda$, by (2.3), since $W_1(\phi,\overline{\phi}) = 0$. Hence, if $\operatorname{Im}\lambda > 0$, the exterior of the circle C_a corresponds to $\operatorname{Im} z < 0$ and the interior corresponds to $\operatorname{Im} z > 0$.

Since

$$\begin{aligned} -\operatorname{Im} z = \frac{i}{2}(z - \overline{z}) &= \frac{i}{2}\left(-\frac{l\phi'(a) + \theta'(a)}{l\phi(a) + \theta(a)} + \overline{\left(\frac{l\phi'(a) + \theta'(a)}{l\phi(a) + \theta(a)}\right)}\right) \\ &= \frac{i}{2|l\phi(a) + \theta(a)|^2}\left(|l|^2 W_a(\phi,\overline{\phi}) + W_a(\theta,\overline{\theta}) + lW_a(\phi,\overline{\theta}) + \overline{l}W_a(\theta,\overline{\phi})\right) \\ &= \frac{iW_a(\theta + l\phi, \overline{\theta} + \overline{l\phi})}{2|l\phi(a) + \theta(a)|^2}, \end{aligned} \tag{2.7}$$

we have, by (2.3), that $\operatorname{Im} z > 0$ if and only if

$$2\operatorname{Im}\lambda \int_a^1 |\theta + l\phi|^2 dx < -iW_1(\theta + l\phi, \overline{\theta} + \overline{l\phi}) = 2\operatorname{Im} l.$$

Hence, in case $\operatorname{Im}\lambda > 0$, l is inside C_a iff

$$\int_a^1 |\theta + l\phi|^2 dx < \frac{\operatorname{Im} l}{\operatorname{Im}\lambda}.$$

The same holds for $\operatorname{Im}\lambda < 0$. In each case $\operatorname{Im} l$ has the same sign as $\operatorname{Im}\lambda$. It follows that, if l is inside C_a and $0 < a < a' < 1$, then

$$\int_{a'}^{1} |\theta + l\phi|^2 dx < \int_{a}^{1} |\theta + l\phi|^2 dx < \frac{\operatorname{Im} l}{\operatorname{Im}\lambda}.$$

Hence l is also inside $C_{a'}$. Therefore, $C_{a'}$ contains C_a if $a < a'$. It follows that, as $a \searrow 0$, the circle C_a shrinks and thus it converges either to a circle (a.k.a. limit-circle) or to a single point (a.k.a. limit-point).

If $m = m(\lambda)$ is the limit-point, or any point on the limit-circle,

$$\int_{a}^{1} |\theta + l\phi|^2 dx < \frac{\operatorname{Im} m}{\operatorname{Im}\lambda}$$

for all values of a. Hence

$$\int_{0}^{1} |\theta + l\phi|^2 dx < \frac{\operatorname{Im} m}{\operatorname{Im}\lambda}.$$

It follows that for every nonreal of λ , (1.1) has a solution

$$\psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\phi(x,\lambda)$$

belonging to $L_2(0,1)$.

Next lemma was proved in [19]. We give its proof for sake of completeness.

Lemma 2.1. *For real β, $l = l(\lambda)$ is a meromorphic function of λ, with simple poles on the real axis, and thus $m(\lambda)$ is an analytic function in the upper and the lower halfplanes of the λ-plane.*

Proof. By (2.6) the poles of $l(\lambda)$ are the zeros of

$$\phi(a,\lambda)\cos\beta + \phi'(a,\lambda)\sin\beta.$$

But this zeros are eigenvalues of the following problem:

$$\begin{cases} -y'' + q(x)y = \lambda y, \quad x \in [a,1], \\ y(1,\lambda)\cos\alpha + y'(1,\lambda)\sin\alpha = 0, \\ y(a,\lambda)\cos\beta + y'(a,\lambda)\sin\beta = 0. \end{cases}$$

Hence this poles are simple.

The above argument shows that, for a fixed λ, the region of the l-plane inside C_a shrinks as a decreases. Hence $l(\lambda) = l(\lambda, a, \beta)$ is uniformly bounded in any compact region lying entirely in the upper or lower halfplane. Then, in the limit-circle case, there exist sequences $\{a_k\}_{k=1}^{\infty}$ and $\{\beta_k\}_{k=1}^{\infty}$ such that the limit

$$\lim_{k\to\infty} l(\lambda, a_k, \beta_k) = m(\lambda)$$

exists in each half of λ-plane and this limit is an analytic function. In the limit-point case, $l(\lambda)$ has an unique limit $m(\lambda)$, which is an analytic function. □

The reasoning above means that the operator $H_0 = -\frac{d^2}{dx^2} + q(x)$, with boundary condition (2.4) and the zero condition at 0, is not essentially selfadjoint, since the equation $H_0^* u - \lambda u = 0$ has nontrivial L^2 solutions for nonreal λ: in fact, the operator without any boundary condition at 0 is exactly the adjoint operator H_0^*. The space of these solutions is one-dimensional (like any space of solutions of a second-order differential equation with one boundary condition set). Thus, in order to obtain a self-adjoint operator, we need to set a boundary condition at the point $x = 0$. This condition determines the way how the solutions of the equation behave as approaching the left endpoint of our interval.

Similar to the case of the singular point at infinity, considered, for example, in [12, Theorem 2.3.2], this boundary condition has the form

$$\lim_{x\to 0} W\{f, E_\lambda\} = 0, \tag{2.8}$$

where $E_\lambda(x) = \int_{0+}^{\lambda} \varphi(x,\lambda')d\rho(\lambda')$ and $\rho(\lambda)$ is the spectral function of the operator in Weyl sense.

This boundary condition describes the asymptotic oscillation of the solutions when approaching the endpoint. It is hard to be expressed explicitly – in fact, we do not need an explicit expression for this condition. Note, however the paper [8], where for a problem of the type we consider, a physical meaning is assigned to this boundary condition.

3. PROPERTIES OF THE m-FUNCTION AND THE NATURE OF THE SPECTRUM FOR A POTENTIAL WITH STRONG NEGATIVE SINGULARITY AT THE ENDPOINT

Now we formulate the general conditions imposed on the potential $q(x)$ and derive some properties of the m-function for this case.

Lemma 3.1. *Let $q(x)$ be a differentiable function on $(0,1)$ such that $q'(x) > 0$, $q(x) \to -\infty$ as $x \to 0$, $q''(x)$ is ultimately of one sign and*

$$q'(x) = O(|q(x)|)^c, \tag{3.1}$$

for $0 < c < \frac{3}{2}$. Then $m(\lambda)$ is a meromorphic function with simple poles on the real axis, and the whole spectrum of the problem (1.1) is discrete.

Proof. Let $\phi(x) = \phi(x,\lambda)$ be the solution of (1.1) which satisfies the boundary conditions

$$\phi(1) = \sin\alpha, \quad \phi'(1) = -\cos\alpha.$$

We set

$$\xi(x,\lambda) := \int_x^1 (\lambda - q(t))^{\frac{1}{2}} dt, \ p(x,\lambda) = (\lambda - q(x))^{-\frac{1}{4}},$$

$$R(x,\lambda) := -\frac{q''(x)}{4(\lambda - q(x))^{\frac{3}{2}}} - \frac{5(q'(x))^2}{16(\lambda - q(x))^{\frac{5}{2}}}.$$

The expression for $R(x,\lambda)$ occurs naturally from the basic equation (1.1) (see [19, 5.8]). Let ρ be a real constant such that $\rho - q(x) > 0$ for $0 < x \leq 1$. Then

$$\begin{aligned}
&\int_x^1 p(t,\rho)(q(t)-\lambda)\phi(t)\sin(\xi(x,\rho)-\xi(t,\rho))dt \\
&= \int_x^1 p(t,\rho)\phi''(t)\sin(\xi(x,\rho)-\xi(t,\rho))dt = p(1,\rho)\phi'(1)\sin\xi(x,\rho) \\
&\quad - \int_x^1 \phi'(t)\left(p'(t,\rho)\sin(\xi(x,\rho)-\xi(t,\rho)) + p(t,\rho)\cos(\xi(x,\rho)-\xi(t,\rho))(\rho-q(t))^{\frac{1}{2}}\right)dt \\
&= -p(1,\rho)\cos\alpha\sin\xi(x,\rho) + \phi(x)(\rho-q(x))^{\frac{1}{4}} - p'(1,\rho)\sin\alpha\sin\xi(x,\rho) \\
&\quad - \sin\alpha(\rho-q(1))^{\frac{1}{4}}\cos\xi(x,\rho) \\
&\quad + \int_x^1 \phi(t)\Big(p''(t,\rho) - p(t,\rho)(\rho-q(t))\Big)\sin(\xi(x,\rho)-\xi(t,\rho))dt \\
&\quad + \int_x^1 \phi(t)\Big(2p'(t,\rho)(\rho-q(t))^{\frac{1}{2}} - \frac{1}{2}p(t,\rho)\frac{q'(t)}{(\rho-q(t))^{\frac{1}{2}}}\Big)\cos(\xi(x,\rho)-\xi(t,\rho))dt.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\phi(x)(\rho-q(x))^{\frac{1}{4}} \\
&= \left(\frac{\cos\alpha}{(\rho-q(1))^{\frac{1}{4}}} + \frac{1}{4}\frac{q'(1)\sin\alpha}{(\rho-q(1))^{\frac{5}{4}}}\right)\sin\xi(x,\rho) + (\rho-q(1))^{\frac{1}{4}}\sin\alpha\cos\xi(x,\rho) \\
&\quad + \int_x^1 \phi(t)(\rho-q(t))^{\frac{1}{4}}\left(\frac{\rho-\lambda}{(\rho-q(t))^{\frac{1}{2}}} - R(t,\rho)\right)\sin(\xi(x,\rho)-\xi(t,\rho))dt.
\end{aligned} \tag{3.2}$$

Now we define the functions

$$\phi_1(x) = \phi(x)(\rho-q(x))^{\frac{1}{4}},\ R_1(t) = \frac{\rho-\lambda}{(\rho-q(t))^{\frac{1}{2}}} - R(t,\rho).$$

Note that

$$\int_0^1 |R_1(t)|dt < \infty.$$

Indeed, from (3.1) it follows

$$\int_0^1 \frac{(q'(t))^2 dt}{(\rho - q(t))^{\frac{5}{2}}} = \int_0^1 \frac{q'(t)}{(\rho - q(t))^c} \frac{q'(t)dt}{(\rho - q(t))^{\frac{5}{2}-c}}$$
$$\leq C_\rho \int_0^1 \frac{q'(t)dt}{(\rho - q(t))^{\frac{5}{2}-c}} = C_\rho \int_{q(1)}^{+\infty} \frac{ds}{(\rho - s)^{\frac{5}{2}-c}} < \infty,$$

where C_ρ depends on ρ only, and

$$\int_0^1 \frac{q''(t)dt}{(\rho - q(t))^{\frac{3}{2}}} = \left[\frac{q'(t)dt}{(\rho - q(t))^{\frac{3}{2}}}\right]_0^1 + \frac{3}{2}\int_0^1 \frac{(q'(t))^2 dt}{(\rho - q(t))^{\frac{5}{2}}} < \infty.$$

In terms of the new functions, (3.2) can be written as

$$\phi_1(x) = A\cos\xi(x,\rho) + B\sin\xi(x,\rho) + \int_x^1 \phi_1(t)R_1(t)\sin(\xi(x,\rho) - \xi(t,\rho))dt.$$

By Grönwall's inequality, it follows that $\phi_1(x)$ is bounded and

$$\phi_1(x) = A\cos\xi(x,\rho) + B\sin\xi(x,\rho) + \int_0^1 \phi_1(t)R_1(t)\sin(\xi(x,\rho) - \xi(t,\rho))dt + o(1), \tag{3.3}$$

where A and B are independent of λ, and the integral converges uniformly over any bounded λ-region, and therefore represents an entire analytic function of λ variable. We have, therefore, as $x \to 0$,

$$\phi(x)(\rho - q(x))^{\frac{1}{4}} = \gamma(\lambda)\cos\xi(x,\rho) + \delta(\lambda)\sin\xi(x,\rho) + o(1), \tag{3.4}$$

where $\gamma(\lambda)$ and $\delta(\lambda)$ are entire functions of λ.

Similarly, by using the differentiated form of (3.3) we obtain

$$\phi'(x)(\rho - q(x))^{-\frac{1}{4}} = -\delta(\lambda)\cos\xi(x,\rho) + \gamma(\lambda)\sin\xi(x,\rho) + o(1). \tag{3.5}$$

Further on, if $\theta(x,\lambda)$ is the solution of (1.1) such that

$$\theta(1,\lambda) = \cos\alpha, \quad \theta'(1,\lambda) = \sin\alpha,$$

we have

$$\theta(x)(\rho - q(x))^{\frac{1}{4}} = \gamma_1(\lambda)\cos\xi(x,\rho) + \delta_1(\lambda)\sin\xi(x,\rho) + o(1), \tag{3.6}$$

$$\theta'(x)(\rho - q(x))^{-\frac{1}{4}} = -\delta_1(\lambda)\cos\xi(x,\rho) + \gamma_1(\lambda)\sin\xi(x,\rho) + o(1), \tag{3.7}$$

where $\gamma_1(\lambda)$ and $\delta_1(\lambda)$ are also entire analytic functions of λ.

Again we note that $\theta(x,\lambda)$ and $\phi(x,\lambda)$ both belong to $L_2(0,1)$ for all values of λ, so we are in Weyl's limit-circle case.

Substituting (3.4)-(3.7) into the right hand side of (2.6), and setting

$$\cot\beta = (\rho - q(a))^{\frac{1}{2}} \cot\beta',$$

we obtain

$$l(\lambda) = -\frac{\gamma_1(\lambda)\cos(\xi(a,\rho)-\beta') + \delta_1(\lambda)\sin(\xi(a,\rho)-\beta') + o(1)}{\gamma(\lambda)\cos(\xi(a,\rho)-\beta') + \delta(\lambda)\sin(\xi(a,\rho)-\beta') + o(1)}.$$

We choose β' as a function of a so that $\xi(a,\rho)-\beta' = \kappa$, and obtain

$$l(\lambda) \to -\frac{\gamma_1(\lambda)\cos\kappa + \delta_1(\lambda)\sin\kappa}{\gamma(\lambda)\cos\kappa + \delta(\lambda)\sin\kappa},$$

as a tends to 0. When κ varies, this equation describes a circle, which is the limit-circle in question. Thus, in the notation of the previous section,

$$m(\lambda) = -\frac{\gamma_1(\lambda)\cos\kappa + \delta_1(\lambda)\sin\kappa}{\gamma(\lambda)\cos\kappa + \delta(\lambda)\sin\kappa}.$$

For any κ fixed, which, again, corresponds to fixing the boundary condition at the singular point, this equation describes a meromorphic function of λ variable, and the eigenvalues of the operator with this boundary condition are its poles. In particular, this means that the whole spectrum is discrete. □

4. COUNTING ZEROS OF SOLUTIONS

This is the most technical part of the paper. Therefore, it is proper to explain what is going on, before making complicated calculations.

It is known that for a regular Sturm-Liouville problem on a finite interval, the eigenvalues are counted by counting the number of zeros on this interval of the corresponding eigenfunctions. For our singular problem, each eigenfunction oscillates rapidly near the singular endpoint and has therefore infinitely many zeros. So, the idea, initially implemented by P.Heywood, consists in finding for the eigenfunctions in question rather sharp two-sided estimates for the zeros on an interval $(a,1)$, with $a>0$. This leads to two-sided estimates for the counting functions of the eigenvalues of a regular problem on this finite interval. In the next section we will show that these eigenvalue estimates admit passage to the limit as $a \to 0$.

It is hard to count zeros of eigenfunctions when these eigenfunctions are not known. Therefore we construct some other functions, for which the counting task is somewhat easier, while the number of their zeros differs from the one for the eigenfunctions in a controllable way.

In the previous section we imposed conditions on the potential $q(x)$ granting the limit-circle case at $x=0$ for the equation (1.1). Now, in order to study the asymptotics

of the eigenvalues, we need some additional properties of the potential. These conditions will be expressed in terms of the function $h(x) = -q(x)$.

Let $h(x)$ be a twice-differentiable function on $(0, 1]$ which satisfies the following conditions:

(A) $h(1) = 0$;
(B) $h'(x)$ is negative when $x \in (0, 1)$;
(C) $h''(x)$ is ultimately of one sign;
(D) $h''(x) = O(|h'(x)|)^\gamma$ where $1 < \gamma < \frac{4}{3}$;
(E) $\frac{|h'(x)|}{(h(x))^d}$ increases monotonically for some constant $d < \frac{3}{2}$,

where (C), (D) and (E) hold for x close to zero.

We consider the eigenvalue problem associated with the differential equation

$$\frac{d^2y}{dx^2} + (\lambda + h(x))y = 0, \tag{4.1}$$

in the interval $(0, 1)$, and the boundary condition

$$y(1)\cos\alpha + y'(1)\sin\alpha = 0. \tag{4.2}$$

Note that condition (D) implies

$$|h'(x)| = O(h(x))^c, \tag{4.3}$$

where $c = \frac{1}{2-\gamma} < \frac{3}{2}$. Indeed,

$$\begin{aligned} |h'(x)| &= \left| \int_x^1 h''(t)dt - h'(1) \right| = O\left(\int_x^1 |h'(t)|^\gamma dt \right) \\ &= O\left(|h'(x)|^{\gamma-1} \int_x^1 |h'(t)|dt \right) = O\left(|h'(x)|^{\gamma-1} h(x) \right). \end{aligned}$$

Hence

$$|h'(x)|^{2-\gamma} = O(h(x)).$$

Let $\phi(x, \lambda)$ be the solution of (4.1) such that

$$\phi(1, \lambda) = \sin\alpha, \quad \phi'(1, \lambda) = -\cos\alpha. \tag{4.4}$$

First we find the number of zeros of $\phi(x, \lambda)$ on $(a, 1)$ when λ is positive and sufficiently large. For this, we need a technical lemma.

Lemma 4.1. *For λ sufficiently large, the integrals*

$$A = \int_0^1 \frac{(h'(x))^2 dx}{(\lambda + h(x))^{\frac{5}{2}}}, \quad B = \int_0^1 \frac{h''(x)dx}{(\lambda + h(x))^{\frac{3}{2}}}$$

converge and are bounded uniformly in λ.

Proof. Fix $X \in (0,1)$. Since the function $\frac{(h'(x))^2}{(\lambda+h(x))^{\frac{5}{2}}}$ is continuous and decreasing,

$$A = \int_0^X \frac{(h'(x))^2 dx}{(\lambda+h(x))^{\frac{5}{2}}} + O(1) = O\left(\int_0^X \frac{|h'(x)|(h(x))^c dx}{(\lambda+h(x))^{\frac{5}{2}}}\right) + O(1).$$

We make the change of the variable $h(x) = \lambda t$, and then

$$A = O\left(\int_{\frac{h(X)}{\lambda}}^{+\infty} \frac{\lambda(\lambda t)^c dt}{(\lambda+\lambda t)^{\frac{5}{2}}}\right) + O(1) = \lambda^{c-\frac{3}{2}} O\left(\int_{\frac{h(X)}{\lambda}}^{+\infty} \frac{t^c dt}{(1+t)^{\frac{5}{2}}}\right) + O(1)$$
$$\leq \lambda^{c-\frac{3}{2}} \int_0^{+\infty} \frac{t^c dt}{(1+t)^{\frac{5}{2}}} + O(1) \leq C_1^{c-\frac{3}{2}} K.$$

Hence, A is bounded for $\lambda > C_1$. By integrating by parts, we have

$$B = \int_0^X \frac{h''(x)dx}{(\lambda+h(x))^{\frac{3}{2}}} + O(1) = \left[\frac{h'(x)}{(\lambda+h(x))^{\frac{3}{2}}}\right]_0^X + \frac{3}{2}\int_0^X \frac{(h'(x))^2 dx}{(\lambda+h(x))^{\frac{5}{2}}} + O(1).$$

Then the second result follows from (4.3) and the first one. □

Now we define the continuous function $\zeta(x)$:

$$\zeta(x) = \tan^{-1}\left(\frac{(\lambda+h(x))^{\frac{1}{2}}\phi(x,\lambda)}{\phi'(x,\lambda)}\right), \quad x \in (0,1],\ \lambda > 0$$

with $0 \leq \zeta(1) \leq \pi$. The continuity condition governs the choice of the branch of $\tan^{-1}$ at the points where $\phi' = 0$. We differentiate to obtain

$$\zeta'(x) = (\lambda+h(x))^{\frac{1}{2}} + \frac{1}{4}h'(x)(\lambda+h(x))^{-1}\sin 2\zeta$$
$$= (\lambda+h(x))^{\frac{1}{2}} + \frac{1}{4}h'(x)(\lambda+h(x))^{-\frac{3}{2}}\sin 2\zeta\Big(\zeta'(x) - \frac{1}{4}h'(x)(\lambda+h(x))^{-1}\sin 2\zeta\Big), \tag{4.5}$$

and the integration gives

$$\zeta(1) - \zeta(a) = \int_a^1 (\lambda+h(x))^{\frac{1}{2}} dx + \frac{1}{4}I_1 - \frac{1}{16}I_2, \tag{4.6}$$

where

$$I_1 = \int_a^1 \frac{h'(x)\zeta'(x)\sin 2\zeta dx}{(\lambda+h(x))^{\frac{3}{2}}},\ I_2 = \int_a^1 \frac{(h'(x))^2\sin^2 2\zeta dx}{(\lambda+h(x))^{\frac{5}{2}}}.$$

Now, by Lemma 4.1, I_2 is bounded for $\lambda > C_1$. We integrate by parts:

$$I_1 = \left[-\frac{\frac{1}{2}h'(x)\cos(2\zeta)}{(\lambda + h(x))^{\frac{3}{2}}} \right]_a^1 + \frac{1}{2}\int_a^1 \left(\frac{h''(x)}{(\lambda + h(x))^{\frac{3}{2}}} - \frac{\frac{3}{2}(h'(x))^2}{(\lambda + h(x))^{\frac{5}{2}}} \right) \cos 2\zeta dx.$$

The integral term is bounded at the upper limit since $\lambda > C_1$ and at the lower limit since $h'(x) = o(h(x))^{\frac{3}{2}}$. Hence, by Lemma 4.1, I_1 is bounded for $\lambda > C_1$. So, we have

$$\zeta(1) - \zeta(a) = \int_a^1 (\lambda + h(x))^{\frac{1}{2}} dx + O(1), \tag{4.7}$$

where the error term is bounded for $\lambda > C_1$ and for all $a > 0$.

We write (4.5) in the form

$$\zeta'(x) = (\lambda + h(x))^{\frac{1}{2}} \left(1 + \frac{\frac{1}{4}h'(x)\sin 2\zeta}{(\lambda + h(x))^{\frac{3}{2}}} \right).$$

Since $|h'(x)| = O(h(x))^{\frac{3}{2}}$, we have $h'(x)(h(x))^{-\frac{3}{2}} < 1$, say, for $x < A$. But $h'(x)$ is continuous on $(0, 1]$, and it is therefore bounded on $[A, 1]$. Then for sufficiently large λ, say, greater than C_2, we have $\zeta'(x) > 0$. Therefore $\zeta(x)$ is monotonic increasing.

Since $\phi(x, \lambda)$ is a non-trivial solution of (4.1), ϕ and ϕ' cannot vanish simultaneously. Therefore, zeros of $\phi(x, \lambda)$ occur if and only if $\zeta(x)$ is equal to an integer multiple of π. Let N_1 be the number of zeros of $\phi(x, \lambda)$ on the interval $(a, 1)$. Then, by (4.7),

$$N_1 = \frac{1}{\pi}\int_a^1 (\lambda + h(x))^{\frac{1}{2}} dx + O(1). \tag{4.8}$$

Next we consider negative values of λ and write $\mu = -\lambda$.

Lemma 4.2. *For $h(x)$ as above, we have $h'(x) \to -\infty$ as $x \to 0$ and $h''(x) > 0$ for $x > 0$.*

Proof. By conditions (B) and (C), we know that $h'(x)$ is negative and monotonic on $(0, X)$, $X \leq 1$. Hence, as x tends to zero, $h'(x)$ tends to minus infinity or to a finite limit. But, if it had finite limit, $|h'(x)|$ would be less than some bound M. On the other hand, the mean value theorem gives, for some $\alpha \in [x, X]$,

$$h(x) = h(X) + h'(\alpha)(x - X) \leq h(X) - M(x - X).$$

The left-hand side tends to infinity as $x \to 0$, while the right-hand side is bounded. So $h'(x)$ must tend to minus infinity when x approaches zero. Additionally, this reasoning implies that $h''(x) > 0$ for x sufficiently close to zero, therefore, by (C), it is positive for $x \in (0, 1]$. □

Since the function $h(x)$ decreases monotonically in $(0,1]$ and tends to plus infinity as x tends to zero, we can define for any positive value of μ a unique positive number $p = p(\mu)$ by

$$h(p) = \mu.$$

Now suppose that $\mu \geq h(X)$, so that $p(\mu) \leq X$. Then, by Lemma 4.2, the function $(x-p)(h'(x))^{\frac{1}{3}}$ decreases on $(0,p)$ and $\lim_{x\to 0}(x-p)(h'(x))^{\frac{1}{3}} = +\infty$. Hence there exists a unique point $P = P(\mu)$ such that $P(\mu) < p(\mu)$ and

$$(P-p)(h'(P))^{\frac{1}{3}} = 1. \tag{4.9}$$

By the mean value theorem,

$$h'(p) - h'(P) = (p-P)h''(\alpha),$$

for some $\alpha \in [P,p]$, therefore,

$$h'(p) \leq h'(P) + (p-P)\max_{x\in[P,p]} h''(x),$$

and so, by (D) and (4.9),

$$\begin{aligned} h'(p) &\leq h'(P) + C(p-P)|h'(P)|^{\gamma} \\ &= h'(P) + (p-P)|h'(P)|^{\frac{1}{3}}C|h'(P)|^{\gamma-\frac{1}{3}} = h'(P) + C|h'(P)|^{\gamma-\frac{1}{3}}, \end{aligned}$$

or equivalently,

$$-h'(p) \geq -h'(P)\left(1 - C|h'(P)|^{\gamma-\frac{4}{3}}\right).$$

Since $\gamma < \frac{4}{3}$, the right-hand side tends to $-h'(P)$ when μ goes to infinity. Therefore,

$$-h'(p) \geq -\frac{1}{2}h'(P), \tag{4.10}$$

for $\mu > C_3 > h(X)$, where C_3 is a suitable large constant.

Using (4.9) and (4.10), we have, for $\mu > C_3$,

$$h(P) - \mu = h(P) - h(p) \leq (P-p)h'(P) = (h'(P))^{\frac{2}{3}} \tag{4.11}$$

and

$$h(P) - \mu \geq (P-p)h'(p) \geq -\frac{1}{2}h'(P)(p-P) = \frac{1}{2}(h'(P))^{\frac{2}{3}}. \tag{4.12}$$

Further we need an estimate, similar to the one in Lemma 4.1.

Lemma 4.3. *The integrals*

$$U = \int_0^{P(\mu)} \frac{(h'(x))^2 dx}{(h(x)-\mu)^{\frac{5}{2}}},\ V = \int_0^{P(\mu)} \frac{h''(x)dx}{(h(x)-\mu)^{\frac{3}{2}}}$$

converge and are bounded for $\mu > C_3$.

Proof. From (E) we know that $\frac{-h'(x)}{(h(x))^d}$ increases, where d is a positive number, by Lemma 4.2. Hence, the function

$$-\frac{h'(x)}{(h(x)-\mu)^d} = \frac{-h'(x)}{(h(x))^d}\cdot\frac{1}{(1-\frac{\mu}{h(x)})^d}$$

is monotonically increasing on $(0, p(\mu)]$, provided $\mu > h(X)$. Therefore, if $\mu > C_3$,

$$U \le \frac{-h'(P)}{(h(P)-\mu)^d}\int\limits_0^{P(\mu)}\frac{-h'(x)dx}{(h(x)-\mu)^{\frac{5}{2}-d}} = \frac{-h'(P)}{(h(P)-\mu)^d}\left(\int\limits_{h(P)}^{+\infty}\frac{dt}{(t-\mu)^{\frac{5}{2}-d}}\right)$$
$$= -\frac{h'(P)}{(h(P)-\mu)^d}\cdot\frac{(h(P)-\mu)^{-\frac{3}{2}+d}}{\frac{3}{2}-d} = -\frac{h'(P)}{(h(P)-\mu)^{\frac{3}{2}}}\cdot\frac{1}{\frac{3}{2}-d}.$$

Further, by (4.12), we have

$$U \le \frac{1}{\frac{3}{2}-d}\cdot\frac{-h'(P)}{(\frac{1}{4})^{\frac{1}{3}}|h'(P)|} = \frac{2\sqrt{2}}{\frac{3}{2}-d}.$$

We integrate V by parts to obtain

$$V = \left[\frac{h'(x)}{(h(x)-\mu)^{\frac{3}{2}}}\right]_0^{P(\mu)} + \frac{3}{2}\int\limits_0^{P(\mu)}\frac{(h'(x))^2dx}{(h(x)-\mu)^{\frac{5}{2}}}.$$

But

$$\frac{-h'(x)}{(h(x)-\mu)^{\frac{3}{2}}} \le \frac{-h'(x)}{(h(x)-\mu)^d} = O(1),$$

when $x \to 0$. Then $V = \frac{3}{2}U + O(1)$ which is bounded. □

Let $\phi(x,\lambda)$ be the solution of (4.1) which satisfies (4.2), and suppose that $\mu > C_3$. We define for positive values of x and λ the continuous function $\eta(x)$ by the formula

$$\eta(x) = \tan^{-1}\left(\frac{(h(x)-\mu)^{\frac{1}{2}}\phi(x,\lambda)}{\phi'(x,\lambda)}\right),$$

where $0 \le \eta(1) \le \pi$. Again, the continuity condition determines the choice of the branch of $\tan^{-1}$. Then

$$\eta'(x) = (h(x)-\mu)^{\frac{1}{2}} + \frac{1}{4}h'(x)(h(x)-\mu)^{-1}\sin 2\eta$$
$$= (h(x)-\mu)^{\frac{1}{2}} + \frac{1}{4}h'(x)(h(x)-\mu)^{-\frac{3}{2}}\sin 2\eta\big(\eta'(x) - \frac{1}{4}h'(x)(h(x)-\mu)^{-1}\sin 2\eta\big). \tag{4.13}$$

The integration gives

$$\eta(P) - \eta(a) = \int_a^P (h(x) - \mu)^{\frac{1}{2}} dx + \frac{1}{4} I_3 - \frac{1}{16} I_4, \tag{4.14}$$

where, for $a < P$,

$$I_3 = \int_a^P \frac{h'(x)\eta'(x) \sin 2\eta dx}{(h(x) - \mu)^{\frac{3}{2}}}, \quad I_4 = \int_a^P \frac{(h'(x))^2 \sin^2 2\eta dx}{(h(x) - \mu)^{\frac{5}{2}}}.$$

By Lemma 4.3, I_4 is bounded, since $\mu > C_3$. We integrate by parts:

$$I_3 = \left[-\frac{\frac{1}{2} h'(x) \cos(2\eta)}{(h(x) - \mu)^{\frac{3}{2}}} \right]_a^P + \frac{1}{2} \int_a^P \left(\frac{h''(x)}{(h(x) - \mu)^{\frac{3}{2}}} - \frac{\frac{3}{2}(h'(x))^2}{(h(x) - \mu)^{\frac{5}{2}}} \right) \cos 2\eta dx.$$

Condition (E) implies that the function

$$-\frac{h'(x)}{(h(x) - \mu)^{\frac{3}{2}}} = \frac{-h'(x)}{(h(x))^d} \cdot \frac{1}{(h(x))^{\frac{3}{2} - d}} \cdot \frac{1}{(1 - \frac{\mu}{h(x)})^{\frac{3}{2}}}$$

is positive and increases on $(0, P(\mu))$. Therefore

$$|I_3| \leq -\frac{h'(P)}{(h(P) - \mu)^{\frac{3}{2}}} + \frac{1}{2} V + \frac{3}{4} U,$$

and I_3 is bounded by lemma 4.3. Combining this with (4.14), we have

$$\eta(P) - \eta(a) = \int_a^P (h(x) - \mu)^{\frac{1}{2}} dx + O(1), \tag{4.15}$$

where the error term is bounded for $\mu > C_3$ and $a < P(\mu)$.

On the other hand, by (4.13),

$$\eta'(x) = (h(x) - \mu)^{\frac{1}{2}} \left(1 + \frac{\frac{1}{4} h'(x) \sin 2\eta}{(h(x) - \mu)^{\frac{3}{2}}} \right).$$

As we noted, the function $\left| h'(x)(h(x) - \mu)^{-\frac{3}{2}} \right|$ is increasing. Hence

$$\left| \frac{\frac{1}{4} h'(x) \sin 2\eta}{(h(x) - \mu)^{\frac{3}{2}}} \right| \leq \left| \frac{\frac{1}{4} h'(x)}{(h(x) - \mu)^{\frac{3}{2}}} \right| \leq \frac{1}{4} \left| \frac{h'(P)}{2\sqrt{2} h'(P)} \right|,$$

(in the last inequality we used (4.12)). So $\eta'(x)$ is positive for $x < P(\mu)$. Then $\eta(x)$ increases for $x < P(\mu)$. As before, it follows from the definition of $\eta(x)$ that

the number N_2 of zeros of $\phi(x,\lambda)$ on the interval $0<a<x\le P(\mu)$, differs at most by an endpoint correction of size one from $(\eta(P)-\eta(a))/\pi$. Hence, by (4.15)

$$N_2=\frac{1}{\pi}\int\limits_a^P (h(x)-\mu)^{\frac12}dx+O(1), \tag{4.16}$$

provided $a<P(\mu)$.

Recall that $\phi(x,\lambda)$ satisfies the differential equation

$$\phi''+(h(x)-\mu)\phi=0,$$

and $h(x)-\mu<h(P)-\mu$ for $x>P(\mu)$. Note also that $\sin[(h(P)-\mu)^{\frac12}x]$ is a solution of the differential equation

$$\phi''+(h(P)-\mu)\phi=0.$$

Now, by Sturm's comparison theorem ([18, (5.9)]), between any two consecutive zeros of $\phi(x,\lambda)$ there exists at least one zero of $\sin[(h(P)-\mu)^{\frac12}x]$. Therefore, the number of zeros of $\phi(x,\lambda)$ in the interval $P(\mu)<x<p(\mu)$ does not exceed $(p-P)(h(P)-\mu)^{\frac12}/\pi+2$. But, by using (4.9) and (4.11), we obtain

$$(p-P)(h(P)-\mu)^{\frac12}\le(p-P)\left((h'(P))^{\frac23}\right)^{\frac12}=-(p-P)(h'(P))^{\frac13}=1 \tag{4.17}$$

Moreover, since $\phi(x,\lambda)$ has at most one zero in the interval $p(\mu)\le x<1$, the number, say N_3, of zeros of $\phi(x,\lambda)$ in the interval $a<x<1$, differs by $O(1)$ from N_2. Since, by (4.17),

$$\int\limits_{P(\mu)}^{p(\mu)}(h(x)-\mu)^{\frac12}dx\le(h(P)-\mu)^{\frac12}(p-P)\le1,$$

we have

$$N_3=\frac{1}{\pi}\int\limits_a^p(h(x)-\mu)^{\frac12}dx+O(1), \tag{4.18}$$

where the error term is bounded provided $\mu>C_3$ and $P(\mu)<p(\mu)$.

Let $N(\lambda,a)$ denote the number of zeros of $\phi(x,\lambda)$ on the interval $0<a<x<1$. We define the function $K(\lambda,a)$ for values of λ and a such that $\lambda+h(a)>0$ by the formulae

$$K(\lambda,a)=\begin{cases}\frac{1}{\pi}\int\limits_a^1(\lambda+h(x))^{\frac12}dx, & \lambda\ge0,\\ \frac{1}{\pi}\int\limits_a^{p(|\lambda|)}(\lambda+h(x))^{\frac12}dx & ,0>\lambda>-h(a).\end{cases}$$

It will be shown now that

$$N(\lambda,a)=K(\lambda,a)+O(1), \tag{4.19}$$

for all positive λ provided $a < P(C_3)$, and for all negative λ provided $a < \min\{P(\mu), P(C_3)\}$, where $\mu = -\lambda$. For $\lambda \geq C_2$ and $\mu \geq C_3$ this result has already been obtained in (4.8) and (4.18).

For $-C_3 \leq \lambda \leq C_2$, we have $h(x) - C_3 \leq h(x) + \lambda \leq h(x) + C_2$. Then Sturm's comparison theorem ([18], (5.9)) shows that

$$N(-C_3, a) - 1 \leq N(\lambda, a) \leq N(C_2, a) + 1.$$

Hence, by (4.19) with $\lambda = -C_3, C_2$, we have, if $-C_3 \leq \lambda \leq C_2$ and $a < P(C_3)$,

$$K(-C_3, a) + O(1) \leq N(\lambda, a) \leq K(C_2, a) + O(1). \tag{4.20}$$

On the other hand, if $-C_3 \leq \lambda \leq C_2$ and $a < P(C_3)$, then

$$K(-C_3, a) \leq K(\lambda, a) \leq K(C_2, a), \tag{4.21}$$

since K is a monotonic function of λ when a is fixed. Combining (4.20) and (4.21) we have

$$\begin{aligned} |N(\lambda, a) - K(\lambda, a)| &\leq K(C_2, a) - K(-C_3, a) + O(1) \\ &= \frac{1}{\pi} \int_a^1 (C_2 + h(x))^{\frac{1}{2}} dx - \frac{1}{\pi} \int_a^{p(C_3)} (-C_3 + h(x))^{\frac{1}{2}} dx + O(1) \\ &= \frac{1}{\pi} \int_{p(C_3)}^1 (C_2 + h(x))^{\frac{1}{2}} dx \\ &\quad + \frac{1}{\pi} \int_a^{p(C_3)} \frac{(C_2 + C_3) dx}{(h(x) + C_2)^{\frac{1}{2}} + (h(x) - C_3)^{\frac{1}{2}}} + O(1). \end{aligned}$$

The latter expression is bounded since $h(x) \to +\infty$ as $x \to 0$. Hence, if $a < P(C_3)$, the difference $|N(\lambda, a) - K(\lambda, a)|$ is bounded for $-C_3 < \lambda < C_2$. Then (4.19) holds.

Now we consider the eigenvalue problem for the equation (4.1) on the interval $a \leq x \leq 1$ with boundary conditions (4.2) and

$$y(a) \cos \beta + y'(a) \sin \beta = 0. \tag{4.22}$$

By [18, (5.11)] there exists an increasing sequence of eigenvalues $\lambda_{0a}, \lambda_{1a}, \lambda_{2a} \ldots$ such that the eigenfunction $\psi_{na}(x)$ associated with the eigenvalue λ_{na} has precisely n zeros on the interval $a \leq x \leq 1$. Also, by [18, (1.9)], $\phi(x, \lambda_{na})$ is a multiple of $\psi_{na}(x)$ and therefore has the same zeros. Thus, by (4.19), if a is positive and small enough, then

$$n = K(\lambda_{na}, b) + O(1).$$

But $K(\lambda, a)$ is monotonic in λ for fixed a. Hence, if $E_a(\lambda)$ denotes the number of non-negative eigenvalues not exceeding λ,

$$E_a(\lambda) = K(\lambda, a) - K(0, a) + O(1) = \frac{1}{\pi} \int_a^1 \left((\lambda + h(x))^{\frac{1}{2}} - (h(x))^{\frac{1}{2}} \right) dx + O(1). \tag{4.23}$$

Similarly, if $F_a(\mu)$ denotes the number of negative eigenvalues not exceeding μ by absolute value,

$$\begin{aligned} F_a(\mu) &= K(0,a) - K(-\mu,a) + O(1) \\ &= \frac{1}{\pi}\int_a^1 (h(x))^{\frac{1}{2}}\,dx - \frac{1}{\pi}\int_a^{p(\mu)} (h(x)-\mu)^{\frac{1}{2}}\,dx + O(1), \end{aligned} \tag{4.24}$$

provided $a < \min\{P(\mu), P(C_3)\}$.

It follows from (4.23) and (4.24) that the number of eigenvalues in any fixed finite λ interval (λ_1, λ_2) is bounded as a tends to zero.

5. DISTRIBUTION OF EIGENVALUES

By Lemma 3.1, in order to describe the distribution of the eigenvalues, we need to study the distribution of the poles of the function $m(\lambda)$.

We fix a positive number Λ. We proved in the previous section that, for the problem on $(a,1)$, the number of eigenvalues in the interval $[0,\Lambda]$ is bounded as a tends to zero. Then there must exist an integer R and an infinite sequence $\{a_k\}_{k=0}^{\infty}$, $a_k \to 0$, such that $E_{a_k}(\Lambda) = R$. We choose the smallest of such R. Denote by $0 \le \lambda_{1,a_k} \le \ldots \le \lambda_{R,a_k} \le \Lambda$ the eigenvalues in $(0,\Lambda)$, corresponding to the problem on $(a_k,1)$.

By compactness, we can find a subsequence of $\{a_k\}_{k=0}^{\infty}$ (we can denote it by a_k again), such that the eigenvalues λ_{i,a_k} converge to some numbers $\lambda_i \in (0,\Lambda)$, for $1 \le i \le R$, as $k \to \infty$.

Proposition 5.1. *The numbers $\{\lambda_i\}_{i=1}^{R}$ are different.*

Proof. From (3.4) we obtain, with fixed λ and $x \to 0$,

$$\psi(x,\lambda) = (\rho + h(x))^{-\frac{1}{4}}(\gamma(\lambda)\cos\xi(x,\rho) + \delta(\lambda)\sin\xi(x,\rho)) + o(1). \tag{5.1}$$

But since $\gamma(\lambda)$ and $\delta(\lambda)$ are entire analytic functions, the integral

$$\int_0^1 |\psi(x,\lambda)|^2 dx \tag{5.2}$$

converges uniformly over any finite λ-interval, so it is continuous in $[0,\Lambda]$. This integral cannot vanish for any λ, therefore it has a positive minimum in $[0,\Lambda]$, say m'. Also, by the uniform convergence, for any positive ϵ, we can find $T = T(\epsilon)$ such that

$$\int_0^T |\psi(x,\lambda)|^2 dx < \epsilon \tag{5.3}$$

in $[0, \Lambda]$. But, since the eigenfunctions are orthogonal, for $m \neq n$,

$$\int_{a_k}^{1} \psi(x, \lambda_{m,a_k})\psi(x, \lambda_{n,a_k})dx = 0 \tag{5.4}$$

and if $a_k < T$, Schwartz's inequality and (5.3) give

$$\left| \int_{a_k}^{T} \psi(x, \lambda_{m,a_k})\psi(x, \lambda_{n,a_k})dx \right| < \epsilon \tag{5.5}$$

From (5.4) and (5.5), we have

$$\left| \int_{T}^{1} \psi(x, \lambda_{m,a_k})\psi(x, \lambda_{n,a_k})dx \right| < \epsilon \tag{5.6}$$

for all $a_k < T$. If it were $\lambda_m = \lambda_n$, then, as $k \to \infty$,

$$\int_{T}^{1} \psi(x, \lambda_{m,a_k})\psi(x, \lambda_{n,a_k})dx \to \int_{T}^{1} (\psi(x, \lambda_n))^2 dx \geq m' - \epsilon$$

by (5.3). This contradicts (5.6), for sufficiently small ϵ. □

To prove the next theorem, we need the following lemma, which was proved in [9, Section 8].

Lemma 5.2. *Let $F(\lambda)$ be an analytic function, regular inside and on the boundary of the square*

$$k - r \leq \operatorname{Re}\lambda \leq k + r,$$
$$-r \leq \operatorname{Im}\lambda \leq r,$$

where k and r are positive, except perhaps for simple poles at the points $\lambda = k \pm r$. Suppose that

$$|F(\lambda)| \leq \frac{M}{|\operatorname{Im}\lambda|}$$

throughout the square, except for the line $\operatorname{Im}\lambda = 0$. Then

$$|F(\lambda)| \leq \frac{3M}{r}$$

for λ with $\operatorname{Re}\lambda = k$ and $-r \leq \operatorname{Im}\lambda \leq r$.

Theorem 5.3. *The function $m(\lambda)$ is regular for $0 < \operatorname{Re}\lambda < \Lambda$, except for simple poles at the points $\lambda = \lambda_r$, $r = 1, \ldots, R$.*

Proof. We know from Section 2, that on the circle C_a

$$\int_a^1 |\theta + l\phi|^2 dx \le \frac{\operatorname{Im} l}{\operatorname{Im} \lambda} \le \left| \frac{l}{\operatorname{Im} \lambda} \right|.$$

On the other hand,

$$\frac{1}{2}|l|^2 \int_a^1 |\phi|^2 dx - \int_a^1 |\theta|^2 dx \le \int_a^1 |\theta + l\phi|^2 dx,$$

and therefore

$$|l| \le \frac{1}{|\operatorname{Im} \lambda| \int\limits_a^1 |\phi|^2 dx} + \left(\frac{2 \int\limits_a^1 |\theta|^2 dx}{\int\limits_a^1 |\phi|^2 dx} + \frac{1}{|\operatorname{Im} \lambda|^2 \left(\int\limits_a^1 |\phi|^2 dx \right)^2} \right)^{\frac{1}{2}}.$$

If $0 < a < 1 - \epsilon$, for fixed $\epsilon > 0$ and $|\operatorname{Im} \lambda|$ is bounded, then we can find constants A and B so that

$$|l(\lambda, a)| < A + \frac{B}{|\operatorname{Im} \lambda|}. \tag{5.7}$$

Moreover, for a fixed $K > 0$, there exists a number C such that

$$|l(\lambda, a)| < \frac{C}{|\operatorname{Im} \lambda|}$$

in $D = \{\lambda \in \mathbf{C} : \ 0 \le \operatorname{Re} \lambda \le \Lambda, \ |\operatorname{Im} \lambda| \le K\}$.

From Section 2, we know that all singularities of $l(\lambda, a_k)$ in D are simple poles at the points λ_{r,a_k}, where $r = 1, \dots, R$. Let d_k be the diameter of the set $\{0, \lambda_{1,a_k}, \dots, \lambda_{R,a_k}, \Lambda\}$. We chose η, such that $\eta < \frac{d}{12}$, where $d = \lim_{k \to +\infty} d_k$. Without lost of generality, we may assume that $\eta < \frac{d_k}{12}$, for all $k \ge 1$. For $\varsigma \in \mathbb{C}$ and $\nu > 0$ we denote by Q_ς^ν the square with sides parallel to the axes and of length ν, and center at the point ς. Then applying lemma 5.2 to the function $l(\lambda, a_k)$ and each of the $2R$ squares $Q^{2\eta}_{\lambda_{r,a_k} \pm \eta}$, $r = 1, \dots, R$, we get that

$$|l(\lambda, a_k)| < \frac{3C}{\eta}, \tag{5.8}$$

for $\operatorname{Re} \lambda = \lambda_{r,a_k} \pm \eta$ and $|\operatorname{Im} \lambda| \le \eta$. Then (5.8) holds in $D \setminus \bigcup_{r=1}^R Q^{2\eta}_{\lambda_{r,a_k}}$. But, since $\lambda_{r,a_k} \to \lambda_r$ as $k \to \infty$, if k is large enough, the function $l(\lambda, a_k)$ is regular in $D \setminus \bigcup_{r=1}^R Q^{2\eta}_{\lambda_r}$, and thus (5.8) holds in $D \setminus \bigcup_{r=1}^R Q^{4\eta}_{\lambda_{r,a_k}}$. Therefore, by Vitali's convergence theorem ([20, (5.21)]), $l(\lambda, a_k)$ tends to $m(\lambda)$ uniformly and

$$|m(\lambda)| \le \frac{3C}{\eta},$$

for $\eta \le \operatorname{Re}\lambda \le \Lambda - \eta$ and $|\operatorname{Im}\lambda| \le K - \eta$, excluding $\bigcup_{r=1}^{R} Q_{\lambda_r}^{6\eta}$. Since η and K are arbitrary, it follows that $m(\lambda)$ is regular for $0 < \operatorname{Re}\lambda < \Lambda$ except perhaps for simple poles at the points $\lambda = \lambda_r$, $r = 1, \dots, R$.

So, it remains to prove that λ_r are simple poles (we have already proved such statements for the poles of the approximating regular problems, but now we need it for the singular problem. Moreover, we must exclude the possibility that λ_r is a regular point.) Let $r_{n,a}$ be the residue of the function $l(\lambda, a)$ at the point $\lambda = \lambda_{n,a}$. Then

$$\int_a^1 (\phi(x, \lambda_{n,a}))^2 dx = \frac{1}{r_{n,a}} \tag{5.9}$$

As before, for $\epsilon > 0$, there exists $T = T(\epsilon)$ such that

$$\int_0^T (\phi(x, \lambda))^2 dx < \epsilon$$

for $\lambda \in [0, \Lambda]$. Having fixed T, we can choose k so that $a_k < T$ and

$$\int_T^1 |(\phi(x, \lambda_n))^2 - (\phi(x, \lambda_{n,a_k}))^2| dx < \epsilon.$$

Then

$$\int_0^1 (\phi(x, \lambda_n))^2 dx - \int_{a_k}^1 (\phi(x, \lambda_{n,a_k}))^2 dx$$

$$\le \int_T^1 |(\phi(x, \lambda_n))^2 - (\phi(x, \lambda_{n,a_k}))^2| dx + \int_0^T (\phi(x, \lambda_n))^2 dx + \int_{a_k}^T (\phi(x, \lambda_{n,a_k}))^2 dx < 3\epsilon.$$

This implies, as k tends to infinity,

$$\int_{a_k}^1 (\phi(x, \lambda_{n,a_k}))^2 dx \to \int_0^1 (\phi(x, \lambda_n))^2 dx$$

or, equivalently,

$$\lim_{k \to \infty} r_{n,a_k} = \left(\int_0^1 (\phi(x, \lambda_n))^2 dx \right)^{-1}.$$

But the integral

$$\int_0^1 (\phi(x, \lambda_n))^2 dx < \infty$$

converges uniformly in $\lambda_n \in$, and so has an upper bound, say, M' . Hence

$$\lim_{k\to\infty} r_{n,a_k} \geq \frac{1}{M'}.$$

Let Γ be a simple, closed contour lying within the rectangle $\eta \leq \operatorname{Re}\lambda \leq \Lambda - \eta$ and $|\operatorname{Im}\lambda| \leq K - \eta$, and surrounding the square $Q^{6\eta}_{\lambda_n}$, but not any other part of $Q^{6\eta}_{\lambda_m}$ with $n \neq m$. Then, as k tends to infinity,

$$\int_\Gamma l(\lambda, a_k)d\lambda \to \int_\Gamma m(\lambda)d\lambda$$

uniformly. Hence

$$\int_\Gamma m(\lambda)d\lambda = \lim_{k\to\infty} r_{n,a_k},$$

which is non-zero as shown above. It follows that the contour must include a singularity of the function $m(\lambda)$, which can only be a simple pole at the point $\lambda = \lambda_n$. □

Now we prove the main theorem of this paper.

Theorem 5.4. *Let $h(x) = -q(x)$ be a twice-differentiable function on $(0, 1]$ which satisfies the following conditions:*

(A) $h(1) = 0$*;*
(B) $h'(x)$ *is negative when* $x \in (0, 1)$*;*
(C) $h''(x)$ *is ultimately of one sign;*
(D) $h''(x) = O(|h'(x)|^\gamma)$ *where* $1 < \gamma < \frac{4}{3}$*;*
(E) $\frac{|h'(x)|}{h(x)^d}$ *increases monotonically for some constant* $d < \frac{3}{2}$*,*

where (C)*,* (D) *and* (E) *apply as x tends to zero. Then*

$$N(H, (0, \lambda)) = \frac{\sqrt{\lambda}}{\pi} + O(1),$$
$$N(H, (-\mu, 0)) = \frac{1}{\pi}\int_{p(\mu)}^{1} (h(x))^{\frac{1}{2}}dx + \int_0^{p(\mu)} [(h(x))^{\frac{1}{2}} - (h(x) - \mu)^{\frac{1}{2}}]dx + O(1),$$

where the remainder terms are bounded for all positive values of λ and μ.

Proof. By Theorem 5.3 we know that one can pass to the limit as $a \to 0$ in the eigenvalue estimates on the interval $(a, 1)$. The latter estimates have been obtained in Section 4. Therefore, for positive eigenvalues,

$$N(H, (0, \lambda)) = R + O(1),$$

here we have a possible error term, since we may miss the eigenvalues $\lambda = 0$ and $\lambda = \Lambda$.

But, by (4.23) we have

$$R = \frac{1}{\pi}\int_0^1 [(\lambda + h(x))^{\frac{1}{2}} - (h(x))^{\frac{1}{2}}]dx + O(1).$$

Then

$$\begin{aligned} N(H,(0,\lambda)) &= \frac{1}{\pi}\int_0^1 [(\lambda + h(x))^{\frac{1}{2}} - (h(x))^{\frac{1}{2}}]dx + O(1) \\ &= \frac{\sqrt{\lambda}}{\pi}\int_0^1 \frac{1}{\left(1+\frac{h(x)}{\lambda}\right)^{\frac{1}{2}} + \left(\frac{h(x)}{\lambda}\right)^{\frac{1}{2}}}dx + O(1). \end{aligned}$$

Note that, the integrand is less than 1 and converges to 1 as λ tends to plus infinity. Therefore, the first result follows by the dominated convergence theorem. The asymptotic formula for $N(H,(-\mu,0))$, follows from similar arguments. □

Next corollary immediately follows from the asymptotic formula for the negative spectrum.

Corollary 5.5. *Let $h(x)$ satisfy all conditions of Theorem 5.4. Then the number of negative eigenvalues is finite if and only if*

$$\int_0^1 h(x)^{\frac{1}{2}}dx < +\infty.$$

Corollary 5.6. *Let $h(x)$ satisfies all condition of Theorem 5.4. Then*

$$N(H,(-\mu,0)) = o(N(H,(0,\mu))),$$

when μ tends to infinity.

Proof. It is enough to show that each summand in the asymptotic formula for the negative spectrum is $o(\sqrt{\mu})$:

$$\frac{1}{\sqrt{\mu}}\int_0^{p(\mu)} [(h(x))^{\frac{1}{2}} - (h(x)-\mu)^{\frac{1}{2}}]dx = \int_0^{p(\mu)} \frac{dx}{\left(\frac{h(x)}{\mu}\right)^{\frac{1}{2}} + \left(\frac{h(x)}{\mu}-1\right)^{\frac{1}{2}}} \le p(\mu) \to 0,$$

when $\mu \to +\infty$.

$$\frac{1}{\sqrt{\mu}}\int_{p(\mu)}^1 (h(x))^{\frac{1}{2}}dx \le \frac{1}{\sqrt{\mu}}\left(\int_{p(\mu)}^1 h(x)dx\right)^{\frac{1}{2}}(1-p(\mu))$$

$$= \left(\frac{\int\limits_{p(\mu)}^{1} h(x)dx}{\mu} \right)^{\frac{1}{2}} = \left(\frac{-h(p)p - \int\limits_{p(\mu)}^{1} xh'(x)dx}{-\int\limits_{p(\mu)}^{1} h'(x)dx} \right)^{\frac{1}{2}} \leq \left(\frac{\int\limits_{p(\mu)}^{1} xh'(x)dx}{\int\limits_{p(\mu)}^{1} h'(x)dx} \right)^{\frac{1}{2}} \to 0,$$

by L'Hopital's rule, when $p(\mu) \to 0$. □

6. REMARKS AND EXAMPLES

The conditions of Theorem 5.4 can be relaxed. We do not give here the exact reasoning, that might be rather lengthy, but just give some hints for the method of proving.

Remark 6.1. The condition (A), $h(1) = 0$ is not essential: the change $q \mapsto q - q(1)$ switches the sign of a finite number of eigenvalues, this keeping the asymptotic formulas.

Remark 6.2. The condition (B) can be relaxed: it is sufficient to require that $h'(x) < 0$ for x sufficiently close to 0. Provided that h is bounded outside a neighborhood of 0, one can change h by a bounded function so that after this change the new h would satisfy (B). The fact that the asymptotic formulas hold follows from the relative compactness of the perturbation of h.

Finally, we calculate the asymptotics of the negative eigenvalues for some interesting examples.

Example 6.3. Let $h(x) = x^{-\alpha}$ with $\alpha > 2$. Then, from Theorem 5.4 we obtain

$$\begin{aligned} N(H, (-\mu, 0)) &= \frac{1}{\pi} \int\limits_{\mu^{-\frac{1}{\alpha}}}^{1} x^{-\frac{\alpha}{2}} dx + \frac{1}{\pi} \int\limits_{0}^{\mu^{-\frac{1}{\alpha}}} \left(x^{-\frac{\alpha}{2}} - \left(x^{-\alpha} - \mu \right)^{\frac{1}{2}} \right) dx + O(1) \\ &= \frac{2}{\pi(\alpha - 2)} \mu^{\frac{1}{2} - \frac{1}{\alpha}} + \frac{\mu^{-\frac{1}{\alpha}}}{\pi} \int\limits_{0}^{1} \left(x^{-\frac{\alpha}{2}} \mu^{\frac{1}{2}} - \left(\mu x^{-\alpha} - \mu \right)^{\frac{1}{2}} \right) dx + O(1) \\ &= \frac{\mu^{\frac{1}{2} - \frac{1}{\alpha}}}{\pi} \left(\frac{2}{\alpha - 2} + \int\limits_{0}^{1} \left(x^{-\frac{\alpha}{2}} - \left(x^{-\alpha} - 1 \right)^{\frac{1}{2}} \right) dx \right) + O(1). \end{aligned}$$

Example 6.4. $h(x) = e^{\frac{k}{x}}$, with $k > 0$. To estimate $N(H, (-\mu, 0))$, we estimate integrals in asymptotic formula. On the one hand,

$$\begin{aligned}\int_0^p \frac{1}{(h(x))^{\frac{1}{2}}} &= \int_0^p e^{-\frac{k}{2x}} dx \\ &= -\int_{-p}^0 \frac{2x^2}{k} de^{\frac{k}{2x}} = \frac{2p^2}{k} e^{-\frac{k}{2p}} + \frac{4}{k}\int_{-p}^0 e^{\frac{k}{2x}} x dx + O(1) \\ &< \frac{2p^2}{k} e^{-\frac{k}{2p}} + \frac{4}{k} p^2 e^{-\frac{k}{2p}} + O(1) = \frac{6p^2}{k} e^{-\frac{k}{2p}} + O(1).\end{aligned}$$

Then

$$\frac{3p^2}{k} e^{-\frac{k}{2p}} + O(1) \le \int_0^p \frac{1}{(e^{\frac{k}{x}})^{\frac{1}{2}} + (e^{\frac{k}{x}} - \mu)^{\frac{1}{2}}} dx \le \frac{6p^2}{k} e^{-\frac{k}{2p}} + O(1). \tag{6.1}$$

On the other hand,

$$\begin{aligned}\int_p^1 (h(x))^{\frac{1}{2}} &= \int_p^1 e^{\frac{k}{2x}} dx = -\int_p^1 \frac{2x^2}{k} de^{\frac{k}{2x}} \\ &= \frac{2p^2}{k} e^{\frac{k}{2p}} + \frac{4}{k}\int_p^1 x e^{\frac{k}{2x}} dx + O(1) \\ &= \frac{2p^2}{k} e^{\frac{k}{2p}} + \frac{p^3}{k} e^{\frac{k}{2p}} + \frac{12}{k^2}\int_p^1 x^2 e^{\frac{k}{2x}} dx + O(1),\end{aligned}$$

and then

$$\frac{2p^2}{k} e^{\frac{k}{2p}} + O(1) \le \int_p^1 e^{\frac{k}{2x}} dx \le \frac{2p^2}{k} e^{\frac{k}{2p}} + \frac{p^3}{k} e^{\frac{k}{2p}} + \frac{12p^2}{k^2} e^{\frac{k}{2p}} + O(1) < Cp^2 e^{\frac{k}{2p}} + O(1)$$

for some constant $C > 0$. Combining with (6.1) and recalling relation $\frac{k}{p} = \ln \mu$, we obtain

$$C_1 \frac{\sqrt{\mu}}{\ln^2 \mu} + O(1) < N(H, (-\mu, 0)) < C_2 \frac{\sqrt{\mu}}{\ln^2 \mu} + O(1).$$

Example 6.5. Let $h(x) = e^{\frac{1}{x^2}}$. Similarly to the previous example, one can show

$$C_1 \frac{\sqrt{\mu}}{\ln \mu \sqrt{\ln \mu}} + O(1) < N(H, (-\mu, 0)) < C_2 \frac{\sqrt{\mu}}{\ln \mu \sqrt{\ln \mu}} + O(1).$$

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