Quantum Mechanical Commutation Relations and Theta Functions

BY

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Introduction. A certain nilpotent Lie group plays an important role in the study by H. Weyl [**13**] of the foundations of quantum mechanics. The same group appeared once more in some recent number-theoretic investigations by A. Weil [**12**], whose explicit purpose was to throw the theta-functions away from those parts of analytic number theory where they have played a predominant role in the hands of Hecke and Siegel (among others), or better to replace them by appropriate group-theoretic constructions.

We would like to reverse the whole process and to show how most of the classical properties of theta functions fit into the general group-theoretic framework. The main point is that, whereas the above quoted group has essentially one *equivalence class* of irreducible unitary representations, there are a manifold of concrete realizations of them. More precisely, they can be represented in many different ways as induced representations, and a generalization of Frobenius' reciprocity law, already apparent in some recent work by I. Gelfand and I. Piateskii-Shapiro [**5**], enables us to compare the different representations. One ought to give better foundations to the results of the two last named authors, and we plan to do it at some later occasion.

The first part of the present work is a brief exposition of the Heisenberg commutation relations, and the Schrödinger's and Fock's realizations of them. We describe also H. Weyl's procedure to convene these commutation relations into the realm of group theory. Our second part is devoted to the detailed study of the Weyl's group and its irreducible representations and sketch the application to the theory of theta-functions. It ought to be a pleasant task to recast the whole theory of theta-functions in this framework, but what we have done is just a modest beginning.

The author extends his warmest thanks to N. Katz who wrote a preliminary version of these notes during the Boulder Conference, and to D. Mumford whose ideas about theta-functions greatly helped him to frame his own results. His debt towards A. Weil is of a more subtle kind, but nonetheless real.

I. Commutation relations

1. **Schrödinger representation.** According to the general postulates of quantum mechanics, to every physical system S there is associated a certain complex

Hilbert space $\mathscr{H}$. Every vector of norm one in $\mathscr{H}$ defines a possible state of S, and two vectors a and b define the same state if and only if there exists a constant ω of modulus one with $b = \omega \cdot a$. Moreover, every physical quantity whose measurement depends upon the observation of S is represented by a certain self-adjoint operator in $\mathscr{H}$, in most cases unbounded.

For instance consider the case where S is a mechanical system with a finite number n of degrees of freedom. Choose n position coordinates $q_1, \cdots, q_n$ and the corresponding momenta $p_1, \cdots, p_n$. We assume that any combination of values of the q_k corresponds to some physical state. In that case, the elements of $\mathscr{H} = \mathscr{H}_n$ are pairs of functions $f(q_1, \cdots, q_n)$ and $\hat{f}(p_1, \cdots, p_n)$, both assumed to be square-integrable and related to each other by the Fourier transformation formulas

$$(1)\quad f(q_1, \cdots, q_n) = h^{-n/2} \int \cdots \int \hat{f}(p_1, \cdots, p_n) \cdot e\left(\frac{p_1 q_1 + \cdots + p_n q_n}{h}\right) dp_1 \cdots dp_n,$$

$$(1')\quad \hat{f}(p_1, \cdots, p_n) = h^{-n/2} \int \cdots \int f(q_1, \cdots, q_n) \cdot e\left(-\frac{p_1 q_1 + \cdots + p_n q_n}{h}\right) dq_1 \cdots dq_n.$$

Here h is Planck's constant and $e(t)$ is an abbreviation for $e^{2\pi i t}$. Of course, each of the functions $f(q_1, \cdots, q_n)$ and $\hat{f}(p_1, \cdots, p_n)$ determines the other and the relations (1) and (1′) are equivalent, but there is some advantage putting f and $\hat{f}$ on the same footing. The scalar product in $\mathscr{H}$ is computed according to the equivalent formulas

$$(2)\quad (f|g) = \int \cdots \int \overline{f(q_1, \cdots, q_n)} \cdot g(q_1, \cdots, q_n)\, dq_1 \cdots dq_n,$$

$$(2')\quad (f|g) = \int \cdots \int \overline{\hat{f}(p_1, \cdots, p_n)} \cdot \hat{g}(p_1, \cdots, p_n)\, dp_1 \cdots dp_n.$$

The operational meaning is the following. Assume that S is in a state corresponding to the pair $(f, \hat{f})$. In an experiment aimed at the determination of the position of S, the most we can do is to assert the existence of a probability distribution in the space of the variables $q_1, \cdots, q_n$ with probability density $|f(q_1, \cdots, q_n)|^2$. Similarly, we have a probability distribution in the momentum space with density $|\hat{f}(p_1, \cdots, p_n)|^2$. These assumptions are compatible with the convention associating self-adjoint operators $\mathbf{q}_k$ to q_k and $\mathbf{p}_k$ to p_k in the following way:[1]

$$(3)\quad (\mathbf{q}_k f)(q_1, \cdots, q_n) = q_k \cdot f(q_1, \cdots, q_n),$$

$$(3')\quad (\widehat{\mathbf{p}_k f})(p_1, \cdots, p_n) = p_k \cdot \hat{f}(p_1, \cdots, p_n).$$

[1] The domain of $\mathbf{q}_k$ consists of square-integrable functions f for which the integral

$$\int \cdots \int q_k^2 |f(q_1, \cdots, q_n)|^2\, dq_1 \cdots dq_n$$

is finite. Similarly for $\mathbf{p}_k$.

Generally speaking, the commutator of two operators A and B in $\mathscr{H}$ is defined by $[A, B] = A \cdot B - B \cdot A$.[2] With the previous definitions, we have now the famous *Heisenberg commutations relations*:

$$(4) \qquad [\boldsymbol{q}_j, \boldsymbol{q}_k] = [\boldsymbol{p}_j, \boldsymbol{p}_k] \subset 0, \qquad [\boldsymbol{p}_j, \boldsymbol{q}_k] \subset \frac{h}{2\pi i} \cdot \delta_{jk},$$

where δ_{jk} is 0 if $j \neq k$ and the identity operator I in case $j = k$.

2. **Fock representation.** Another example of a physical system is an assembly of so-called bosons each of which is capable of n different states $e_1, \cdots, e_n$. For instance, one can consider the photons present in a beam of monochromatic light travelling in a well-defined direction; here there are two states e_1 and e_2 corresponding to two independent states of polarization. For the purpose of clarity, we shall in the subsequent discussion call $e_1, \cdots, e_n$ the polarization states of the bosons.

In this case, the Hilbert space $\mathscr{H}$ has an orthonormal basis $\{u(c_1, \cdots, c_n)\}$ where $(c_1, \cdots, c_n)$ runs over all possible combinations of positive integers.[3] In a state of the assembly (to be contrasted with the polarization states of the individual bosons) described by a vector

$$(5) \qquad f = \sum_{c_1, \cdots, c_n} f(c_1, \cdots, c_n) \cdot u(c_1, \cdots, c_n),$$

one can ascribe the probability $|f(c_1, \cdots, c_n)|^2$ to any combination of c_1 bosons in polarization state $e_1, \cdots, c_n$ bosons in polarization state e_n. This is a *bona fide* probability distribution because

$$(6) \qquad \sum_{c_1, \cdots, c_n} |f(c_1, \cdots, c_n)|^2 = \|f\|^2 = 1.$$

The meaning of $u(c_1, \cdots, c_n)$ is therefore that of a pure state in which we can observe c_k bosons in polarization state e_k for $k = 1, \cdots, n$, and a general state is a mixing of such pure states.

The occupation operators $\boldsymbol{N}_1, \cdots, \boldsymbol{N}_n$ are defined by[4]

$$(7) \qquad \boldsymbol{N}_k \cdot u(c_1, \cdots, c_n) = c_k \cdot u(c_1, \cdots, c_n)$$

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[2] Let A and B be two operators in $\mathscr{H}$ with respective domains $\mathscr{D}_A$ and $\mathscr{D}_B$. The operators $A \pm B$ are defined on the domain $\mathscr{D}_A \cap \mathscr{D}_B$ by $(A \pm B) \cdot a = A \cdot a \pm B \cdot a$ and the operator $A \cdot B$ is defined by $(A \cdot B) \cdot a = A \cdot (B \cdot a)$ on the domain consisting of those a in $\mathscr{D}_B$ for which $B \cdot a$ lies in $\mathscr{D}_A$. We write $A \subset B$ in case $\mathscr{D}_A \subset \mathscr{D}_B$ and $A \cdot a = B \cdot a$ for every a in $\mathscr{D}_A$.

[3] We consider 0 a positive number!

[4] The domain of $\boldsymbol{N}_k$ consists of the vectors of the form (5) for which $\sum_{c_1 \cdots c_n} c_k^2 |f(c_1, \cdots, c_n)|^2$ is finite. Similarly, the common domain of $\boldsymbol{a}_k$ and $\boldsymbol{a}_k^*$ is defined by the restriction

$$\sum_{c_1 \cdots c_n} c_k |f(c_1, \cdots, c_n)|^2 < +\infty.$$

in accordance with the previous discussion. But an important role is played by the *creation operators* $a_1, \cdots, a_n$ defined by

$$a_k \cdot u(c_1, \cdots, c_n) = (c_k + 1)^{\frac{1}{2}} \cdot u(c_1, \cdots, c_k + 1, \cdots, c_n) \tag{8}$$

and their adjoints, the *annihilation operators* $a_1^*, \cdots, a_n^*$ given by

$$\begin{aligned} a_k^* \cdot u(c_1, \cdots, c_n) &= 0 \qquad \text{if } c_k = 0 \\ &= c_k^{\frac{1}{2}} \cdot u(c_1, \cdots, c_k - 1, \cdots, c_n) \qquad \text{if } c_k \geqq 1. \end{aligned} \tag{9}$$

With these definitions, we have the following commutation relations:

$$[a_j, a_k] = [a_j^*, a_k^*] \subset 0, \qquad [a_j^*, a_k] \subset \delta_{jk}. \tag{10}$$

The role of the creation and annihilation operators is clarified by the following remarks. The vector $\Omega = u(0, \cdots, 0)$ with no bosons present in either polarization state is understandably called the *vacuum*. It is characterized up to a multiplicative constant by the following relations:

$$a_1^* \cdot \Omega = \cdots = a_n^* \cdot \Omega = 0. \tag{11}$$

Moreover we have

$$u(c_1, \cdots, c_n) = a_1^{c_1} \cdots a_n^{c_n} \cdot \Omega/(c_1! \cdots c_n!)^{\frac{1}{2}}. \tag{12}$$

The operators $a_1, \cdots, a_n$ form a commuting family and by (12) the vectors $P(a_1, \cdots, a_n) \cdot \Omega$ where P runs over the polynoms in n variables with complex coefficients form a dense subspace in $\mathscr{H}$. Note also the relations

$$N_k = a_k \cdot a_k^*, \tag{13}$$

$$a_k^* \cdot P(a_1, \cdots, a_n) \cdot \Omega = P_k'(a_1, \cdots, a_n) \cdot \Omega, \tag{14}$$

where P_k' is the k-th partial derivative of P.

3. **Harmonic oscillator.** We shall now relate the two previous constructions. For that purpose choose two real numbers λ, μ such that $h\lambda\mu = \pi$, and define in the space $\mathscr{H}_n$ of the Schrödinger representation operators $a_1, \cdots, a_n$ by

$$a_k = \lambda \cdot q_k - i\mu \cdot p_k \tag{15}$$

for $k = 1, \cdots, n$. From (4), one deduces (10) by an easy computation. By reference to (11), one looks now for solutions of the equations

$$a_1^* \cdot f = \cdots = a_n^* \cdot f = 0 \tag{16}$$

which are easily transformed into the differential system

$$\left(\frac{\partial}{\partial q_k} + 2\lambda^2 q_k\right) \cdot f(q_1, \cdots, q_n) = 0 \qquad (k = 1, \cdots, n). \tag{17}$$

A normalized solution of this system is given by

$$\Omega(q_1, \cdots, q_n) = (\lambda \pi^{-\frac{1}{2}})^n \exp[-\lambda^2(q_1^2 + \cdots + q_n^2)]. \tag{18}$$

If we define the functions $u(c_1, \cdots, c_n)$ by (12), the relation (8) and (9) are satisfied and also (7) if we define N_k to be equal to $\boldsymbol{a}_k \cdot \boldsymbol{a}_k^*$. Moreover, we have

$$u(c_1, \cdots, c_n)(q_1, \cdots, q_n) = H_{c_1}(q_1) \cdots H_{c_n}(q_n) \tag{19}$$

where the normalized Hermite functions $H_c(q)$ are defined as follows:

$$H_c(q) = \frac{(-1)^c}{2^c \lambda^{c-1} \pi^{\frac{1}{4}} (c!)^{\frac{1}{2}}} e^{\lambda^2 q^2} \left(\frac{d}{dq}\right)^c (e^{-2\lambda^2 q^2}). \tag{20}$$

From the properties of orthonormal polynomials, one deduces that the functions $u(c_1, \cdots, c_n)$ form an orthonormal basis in the space of square-integrable functions of n real variables $q_1, \cdots, q_n$. Otherwise stated, *the Schrödinger and Fock representations are equivalent.*

The physical meaning of this equivalence is depicted by the theory of the harmonic oscillator. According to Newton's mechanics, a particle of mass m bound to a straight line with coordinate q subjected to a force $-K \cdot q$ oscillates sinusoidally with frequency $\nu = (1/2\pi)(K/m)^{\frac{1}{2}}$; the momentum p is $m \cdot v$ where v is the speed and the total energy is

$$E = \frac{p^2}{2m} + \frac{K \cdot q^2}{2}. \tag{21}$$

According to the general quantum-mechanical recipes, we must consider the operator $\boldsymbol{E}$ in $\mathscr{H}_1$ obtained by replacing q by $\boldsymbol{q}$ and p by $\boldsymbol{p}$ in (21). Here the functions $H_0, \cdots, H_c, \cdots$ form an orthonormal basis in $\mathscr{H}_1$ and provided we choose λ according to

$$\lambda = \left(\frac{\pi}{h}\right)^{\frac{1}{2}} (Km)^{\frac{1}{4}} \tag{22}$$

we have $\boldsymbol{E} = h\nu(\boldsymbol{a} \cdot \boldsymbol{a}^* + \frac{1}{2})$, that is

$$\boldsymbol{E} \cdot H_c = (c + \tfrac{1}{2}) \cdot h\nu \cdot H_c \qquad \text{for } c = 0, 1, 2, \cdots. \tag{23}$$

This justifies Planck's initial assumption and can be expressed by saying that a quantum-mechanical harmonic oscillator is equivalent to an assembly of bosons each having one polarization state and energy $h\nu$.[5]

[5] That the vacuum is given the energy $h\nu/2$ is meaningless in view of the fact that *energy differences* only have a definite physical meaning.

4. **Weyl commutation relations.** We shall now transform the Heisenberg commutation relations in a form given first by H. Weyl [**13**]. Consider for that purpose two self-adjoint operators A and B in some Hilbert space $\mathscr{H}$ and the one parameter groups of unitary operators they generate according to Stone's theorem

$$U(s) = e^{isA}, \qquad V(t) = e^{itB}. \tag{24}$$

Assume now that there exists a real constant c such that[6]

$$[A, B] \subset ic \cdot I. \tag{25}$$

If we allow power series expansion of operator exponentials (which is fully justified if A is bounded but not otherwise) and use a well-known formula by Lie

$$e^X \cdot Y \cdot e^{-X} = \sum_{n=0}^{\infty} \frac{1}{n!}[X, [X, \cdots [X, Y] \cdots]], \ n \text{ factors } X \tag{26}$$

we get at once

$$U(s) \cdot B \cdot U(s)^{-1} = B - sc \cdot I. \tag{27}$$

Going to the exponentials in both sides of (27) and multiplying to the right $U(s)$, we obtain

$$U(s) \cdot V(t) = e^{-icst} V(t) \cdot U(s). \tag{28}$$

The steps going from (25) to (28) are fully reversible and the Heisenberg-like commutation relation (25) is *formally* equivalent to the Weyl-like commutation relation (28).

The previous "proof" is open to some criticism and much pain has been devoted to fulfill the gaps. While the equivalence of (27) and (28) makes no difficulty, it appears hard to justify the use of Lie's formula (26) for unbounded A. Rellich [**10**] and Dixmier [**3**] have proved the equivalence of (25) and (28) under the assumption that there exists a dense subspace V of $\mathscr{H}$ contained in the domains of A and B, stable under both A and B, such that the restriction of $A^2 + B^2$ to V be essentially self-adjoint. A general criterion, due to E. Nelson [**8**] and valid for general Lie groups, fully contains the equivalence of (25) and (28) under Rellich-Dixmier assumptions. Another method, used by the author [**2**] and generalized to the case of unbounded operators in Banach spaces by Kato [**6**], rests on the use of Laplace transform and the resolvant formula

$$\int_0^{\infty} e^{-ps} \cdot U(s)\, ds = (p \cdot I - i \cdot A)^{-1} \qquad (p \text{ real} > 0). \tag{29}$$

[6] According to our conventions, this relation means that $[A, B]$ multiply by ic any vector in its domain.

An easy and rigorous argument shows the equivalence of (27) with the relation

$$(30) \qquad (p \cdot I - i \cdot A)^{-1} \cdot B \subset B \cdot (p \cdot I - i \cdot A)^{-1} - c \cdot (p \cdot I - i \cdot A)^{-2}.$$

Right multiplication by $(p \cdot I - i \cdot A)$ gives the fully equivalent relation

$$(31) \qquad (p \cdot I - i \cdot A)^{-1} \cdot B \cdot (p \cdot I - i \cdot A) \subset B - c \cdot (p \cdot I - i \cdot A)^{-1}$$

from which one gets easily the following criterion: *The relation* (28) *holds if and only if* (25) *holds and the domain of* $B \cdot (p \cdot I - i \cdot A)$ *is contained in the domain of* $A \cdot B$ *for every* $p > 0$. It has been shown by Kato [**6**] that the last condition needs only to hold for one value of p.

We give now the Weyl form of the Heisenberg commutation relations (4). Using the fact that two self-adjoint operators commute if and only if their associated one-parameter groups commute, and replacing the relations $[\boldsymbol{p}_k, \boldsymbol{q}_k] \subset (h/2\pi i) \cdot I$ by their Weyl analogue, we obtain

$$(32) \qquad W(t, s, u) \cdot W(t', s', u') = W(t + t' + s' \cdot u, s + s', u + u').$$

Here we used the definition

$$W(t, s, u) = e\left(\frac{t}{h}\right) \cdot e\left(\frac{s_1 \boldsymbol{q}_1}{h}\right) \cdots e\left(\frac{s_n \boldsymbol{q}_n}{h}\right) \cdot e\left(\frac{u_1 \boldsymbol{p}_1}{h}\right) \cdots e\left(\frac{u_n \boldsymbol{p}_n}{h}\right)$$

for t real and two real n-vectors $s = (s_1, \cdots, s_n)$ and $u = (u_1, \cdots, u_n)$; moreover $s \cdot u$ is the scalar product $s_1 u_1 + \cdots + s_n u_n$.

5. **Uniqueness of the representation of commutation relations.** The problem of uniqueness of the representation for the Heisenberg commutation relations can be formulated as follows:

Let be given in some Hilbert space $\mathscr{H}'$ *a family of self-adjoint operators* $\boldsymbol{q}'_1, \cdots, \boldsymbol{q}'_n, \boldsymbol{p}'_1, \cdots, \boldsymbol{p}'_n$ *such that*

$$(33) \qquad [\boldsymbol{q}'_j, \boldsymbol{q}'_k] = [\boldsymbol{p}'_j, \boldsymbol{p}'_k] \subset 0, \qquad [\boldsymbol{p}'_j, \boldsymbol{q}'_k] \subset \frac{h}{2\pi i}\delta_{jk}.$$

Assume that these operators share with the operators in Schrödinger representation the irreducibility property, viz. no closed subspace of $\mathscr{H}'$ *distinct from* 0 *and* $\mathscr{H}'$ *itself reduces simultaneously the operators* $\boldsymbol{q}'_j$ *and* $\boldsymbol{p}'_j$. *Does there exist an isometry* U *of* $\mathscr{H}'$ *onto* $\mathscr{H}$ *such that*

$$(34) \qquad U \cdot \boldsymbol{q}'_j \cdot U^{-1} = \boldsymbol{q}_j, \qquad U \cdot \boldsymbol{p}'_j \cdot U^{-1} = \boldsymbol{p}_j \qquad (j = 1, \cdots, n)?$$

As appropriate counter-examples show, the answer may be negative.[7] The known proofs that uniqueness holds indeed under suitable auxiliary assumptions

[7] For instance, let $\mathscr{H}'$ be the space of square-integrable functions on the closed interval [0, 1] and let $\boldsymbol{q}'$ be the bounded operator defined by $(\boldsymbol{q}' \cdot f)(x) = x \cdot f(x)$ for $0 \leqq x \leqq 1$. Let ω be a complex number of modulus one and define $\boldsymbol{p}'$ as the differential operator $(h/2\pi i)(d/dx)$ with domain the set of absolutely continuous functions f with square-integrable derivative satisfying the boundary condition $f(1) = \omega \cdot f(0)$.

proceed by reduction to the uniqueness problem for Weyl commutation relations. To formulate this problem, we first remark that in the Schrödinger representation we have

$$W(t, s, u) \cdot f(q) = e\left(\frac{t + s \cdot q}{h}\right) \cdot f(q + u) \tag{35}$$

with vector notations, and this in turn implies (32). Moreover, the group law

$$(t, s, u) \cdot (t', s', u') = (t + t' + s' \cdot u, s + s', u + u') \tag{36}$$

makes a real Lie group G out of the real $(2n + 1)$-space.

J. von Neumann [9] and M. Stone [11] have simultaneously proved the following uniqueness theorem:

Any two irreducible unitary representations of the group G, mapping $(t, 0, 0)$ onto the operator $e(t/h) \cdot I$ are unitarily equivalent.

This result solves completely the uniqueness problem for Weyl commutation relations.

II. A certain group and its representations

6. **Description of the group** G. We begin by giving a more invariant description of the Weyl's group. We consider a real finite-dimensional vector space V equipped with a nondegenerate alternating bilinear form B on $V \times V$. The assumptions imply that the dimension of V is an even number $2n$.

The group G is the set of pairs (t, v) where t is a real number and v a vector in V, together with the multiplication law

$$(t, v) \cdot (t', v') = (t + t' + \tfrac{1}{2}B(v, v'), v + v'). \tag{37}$$

The one-parameter subgroups in G are given by[8]

$$g_{t,v}(\lambda) = (\lambda t, \lambda \cdot v) \qquad (\lambda \text{ in } \boldsymbol{R}). \tag{38}$$

It follows for instance that the unit element in G is $e = (0, 0)$ and the inverse of (t, v) is $(-t, -v)$. The Lie algebra of G shall be denoted by $\mathfrak{g}$; according to (38) the vector space $\mathfrak{g}$ is the direct product $\boldsymbol{R} \times V$. We imbed V in $\mathfrak{g}$ by identifying v with $(0, v)$ for any v in V, and we denote by $\mathfrak{z}$ the one-dimensional subspace of $\mathfrak{g}$ generated by $z = (1, 0)$; therefore $\mathfrak{g}$ is the direct sum of $\mathfrak{z}$ and V. Moreover, according to general recipes, we get the bracket in $\mathfrak{g}$ by antisymmetrizing the bilinear terms in the group law (37), that is $[(t, v), (t', v')] = (B(v, v'), 0)$, or with the previous conventions

$$[z, v] = 0, \qquad [v, v'] = B(v, v') \cdot z \tag{39}$$

for v, v' in V. Since B is assumed to be nondegenerate, $\mathfrak{z}$ is the center of $\mathfrak{g}$.

[8] We use standard notations: $\boldsymbol{R}$ is the field of real numbers and $\boldsymbol{C}$ that of complex numbers.

According to (38), the exponential mapping from $\mathfrak{g}$ to G is the identity map of the set $\boldsymbol{R} \times V$. For the sake of clarity, we distinguish between a pair (t, v) considered as an element of $\mathfrak{g}$ or as an element of G. The element $(0, v)$ of G is nothing else than e^v and $(t, 0)$ denoted ι_t or $\iota(t)$ is $e^{t \cdot z}$; more generally, the pair (t, v) as element of G is $\iota_t \cdot e^v$. It is immediate that the group Z image of the homomorphism ι of $\boldsymbol{R}$ into G is both the center and the commutator subgroup of G. By definition of the group law, we get

$$e^v \cdot e^{v'} = \iota(\tfrac{1}{2}B(v, v'))e^{v+v'} \tag{40}$$

for v, v' in V. Finally we have an exact sequence

$$0 \to \boldsymbol{R} \xrightarrow{\iota} G \xrightarrow{\kappa} V \to 0 \tag{41}$$

where κ is given by $\kappa(t, v) = v$.

The characters[9] of Z are given by the formula

$$\chi_\lambda(\iota_t) = e(\lambda t) \tag{42}$$

where λ runs over $\boldsymbol{R}$. The infinitesimal character[9] associated to χ_λ is the linear form on the Lie algebra $\mathfrak{z}$ of Z given by

$$\chi'_\lambda(z) = 2\pi i\lambda. \tag{43}$$

For the purpose of explicit computations, we may introduce a symplectic basis for V with respect to B, let say $\{P_1, \cdots, P_n, Q_1, \cdots, Q_n\}$. We then get a basis $\{z, P_1, \cdots, P_n, Q_1, \cdots, Q_n\}$ of $\mathfrak{g}$ with the property that the only nonzero brackets among basic elements are

$$[P_j, Q_j] = z \qquad (j = 1, \cdots, n). \tag{44}$$

Such a basis of $\mathfrak{g}$ shall be called a *normal basis.*

7. **Infinitesimal representations.** We consider any (unitary) representation $(\pi, \mathscr{H})$ of G. That is, $\mathscr{H}$ is a Hilbert space with scalar product $(a|b)$ linear with respect to b and norm $\|a\| = (a|a)^{\frac{1}{2}}$, and π is a homomorphism of G into the group of unitary operators in $\mathscr{H}$ satisfying the following continuity condition:

(R) *For any pair a, b in $\mathscr{H}$, the function $\phi_{a,b}$ defined on G by $\phi_{a,b}(g) = (a|\pi(g) \cdot b)$ is continuous.*

We let $\mathscr{H}_\infty$ denote the vector subspace in $\mathscr{H}$ consisting of those a's for which $\phi_{a,b}$ is a function of class C^∞ whatever be b in $\mathscr{H}$; the elements in $\mathscr{H}_\infty$ are called *C^∞-vectors.* Among the C^∞-vectors are the vectors

$$\int_G \phi(g)[\pi(g) \cdot a]\, dg \tag{45}$$

[9] A character of a Lie group G is a continuous complex-valued function χ on G such that $|\chi(g)| = 1$ and $\chi(gg') = \chi(g) \cdot \chi(g')$ for g, g' in G. The associated infinitesimal character is the linear form χ' on the Lie algebra $\mathfrak{g}$ of G characterized by $\chi(\exp X) = \exp \chi'(X)$.

where a is any vector in $\mathscr{H}$ and ϕ is a C^∞-function on G with compact support, and the integral is with respect to some Haar measure on G. It has been shown by Gårding [4] that such vectors form a dense set in $\mathscr{H}$, and therefore $\mathscr{H}_\infty$ is dense in $\mathscr{H}$.

For any X in $\mathfrak{g}$, there is a (generally unbounded) operator $\tilde{\pi}(X)$ on $\mathscr{H}$ defined by

$$\tilde{\pi}(X) \cdot a = \lim_{t \to 0} \frac{1}{t} \cdot [\pi(e^{tX}) \cdot a - a] \tag{46}$$

with domain the set of all a's for which the limit exists (strong or weak, it is the same). It can be shown that $\mathscr{H}_\infty$ is the intersection of the domains of all finite products $\tilde{\pi}(X_1) \cdots \tilde{\pi}(X_m)$ where $m > 0$ and $X_1, \cdots, X_m$ run independently over $\mathfrak{g}$.

Let us choose for the moment any basis $\{X_1, \cdots, X_p\}$ of $\mathfrak{g}$ (where $p = 2n + 1$). We define on $\mathscr{H}_\infty$ an increasing sequence of Hilbert norms N_m by

$$N_m(a)^2 = \sum_{|\alpha| \leq m} \|\tilde{\pi}(X_1)^{\alpha_1} \cdots \tilde{\pi}(X_p)^{\alpha_p} \cdot a\|^2 \tag{47}$$

with the standard abbreviations $\alpha = (\alpha_1, \cdots, \alpha_p)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_p$. The norms depend obviously on the chosen basis of $\mathfrak{g}$, but the topology they define on $\mathscr{H}_\infty$ does not; that makes $\mathscr{H}_\infty$ a complete metrizable vector space (an (F)-space). We define $\mathscr{H}_{-\infty}$ as the set of all continuous antilinear[10] forms on $\mathscr{H}_\infty$ and we identify $\mathscr{H}$ with a subspace of $\mathscr{H}_{-\infty}$ by associating to a vector a the antilinear form $b \mapsto (b|a)$ on $\mathscr{H}_\infty$ (note that $\mathscr{H}_\infty$ is dense in $\mathscr{H}$).

It can be shown that the representation of G in $\mathscr{H}$ extends in a natural way to a representation π of G in the (nontopological) vector space $\mathscr{H}_{-\infty}$. Moreover there is a linear representation π' of the Lie algebra $\mathfrak{g}$ in the vector space $\mathscr{H}_{-\infty}$ with the following property: for any X in $\mathfrak{g}$, the domain of $\tilde{\pi}(X)$ is the set of vectors a in $\mathscr{H} \subset \mathscr{H}_{-\infty}$ for which $\pi'(X) \cdot a$ is in $\mathscr{H}$, and we have $\pi'(X) \cdot a = \tilde{\pi}(X) \cdot a$ for such an a. The following relations hold:

$$\pi'((\mathrm{Ad}\, g) \cdot X) = \pi(g) \cdot \pi'(X) \cdot \pi(g)^{-1}, \tag{48}$$

$$(a|\pi'(X) \cdot b) = -(\pi'(X) \cdot a|b) \tag{49}$$

for a, b in $\mathscr{H}_\infty$, for X in $\mathfrak{g}$ and g in G; we denoted by $\mathrm{Ad}\, g$ the automorphism of $\mathfrak{g}$ associated to the inner automorphism $g' \mapsto gg'g^{-1}$ of G. It can be shown that $\mathscr{H}_\infty$ is stable under the operators $\pi(g)$ and $\pi'(X)$.

The previous properties are valid for any representation of any Lie group. They will be considered in detail in the forthcoming paper alluded to in the introduction.[10 bis]

8. **Induced representations.** We recall the classical definition of such representations as given for instance in [1] and [7] under more general circumstances.

[10] A complex valued function F on a complex vector space is called an antilinear form in case the following relations hold $F(v + v') = F(v) + F(v')$ and $F(c \cdot v) = \bar{c} \cdot F(v)$ where $\bar{c}$ is the complex number conjugate to c.

[10 bis] *Added in proof.* L. Schwartz informs me that he defined the spaces $\mathscr{H}_\infty$ and $\mathscr{H}_{-\infty}$ and stated their main properties in his report at the "Second Colloquium on Functional Analysis" held at Liège (Belgium) in May 1966 (see Proceedings, pp. 153–163).

Let χ be a character of some closed subgroup H of G. We let $\mathscr{H}_\chi$ denote the Hilbert space consisting of all functions f on G satisfying the following conditions:

(a) *f is Borel-measurable on G;*

(b) *$f(hg) = \chi(h)\cdot f(g)$ for g in G and h in H;*

(c) *the integral $\int_M |f(g)|^2\, dg$ is finite.*

The norm on $\mathscr{H}_\chi$ is given by

$$\|f\|^2 = \int_M |f(g)|^2\, dg. \tag{50}$$

A few words of explanation are in order. First of all $M = H\backslash G$ is the space of cosets Hg in G. Since G is nilpotent, there exist biinvariant Haar measures dg on G and dh on H, and a measure m on M invariant under the right translations by the elements of G. We abuse the notations by denoting the integral $\int_M \hat{\phi}\, dm$ as $\int_M \phi(g)\, dg$ in case ϕ and $\hat{\phi}$ are related by $\phi(g) = \hat{\phi}(Hg)$. The integral in (50) makes sense because $|\chi(h)| = 1$ implies that $|f|^2$ is constant on every coset Hg by virtue of (b).

To every g in G, there is associated a unitary operator $\pi_\chi(g)$ on $\mathscr{H}_\chi$ by

$$(\pi_\chi(g)\cdot f)(g') = f(g'g) \tag{51}$$

(right translation). The pair $(\pi_\chi, \mathscr{H}_\chi)$ is a representation of G, called the *representation induced by the character χ of H.*[11]

It can be shown that $(\mathscr{H}_\chi)_\infty$ is the set of all C^∞-functions f on G satisfying condition (b) above such that $L\cdot f$ be square-integrable modulo H for every left-invariant differential operator L on G. Accordingly, $(\mathscr{H}_\chi)_{-\infty}$ can be identified with the set of distributions which can be represented as finite sums $\sum_\alpha L_\alpha\cdot f_\alpha$ where the f_α's are in $\mathscr{H}_\chi$ and L_α is a left-invariant differential operator for every α. The representation π' of $\mathfrak{g}$ in $(\mathscr{H}_\chi)_{-\infty}$ is given via the action of the left-invariant vector fields on G. The evaluation map

$$\phi \mapsto \overline{\phi(e)}$$

considered as a functional on $(\mathscr{H}_\chi)_\infty$ is an element u_χ of $(\mathscr{H}_\chi)_{-\infty}$ called the *canonical* one. It can be identified with the distribution on G given by $u_\chi(\phi) = \int_H \phi(h)\chi(h)\, dh$ for every test-function ϕ on G. It satisfies the following equation

$$\pi(h)\cdot u = \chi(h)\cdot u \qquad (h \text{ in } H) \tag{52}$$

which amounts for connected H to be equivalent to the equation

$$\pi'(Y)\cdot u = \chi'(Y)\cdot u \tag{53}$$

for every Y in the Lie algebra of H.

[11] This construction can be expressed in the framework of fibre bundles as follows. On the trivial bundle $G \times C$ over G with fiber C, the group H operates to the left by $h(g, c) = (hg, \chi(h)\cdot c)$ and G operates to the right by $(g, c)\cdot g' = (gg', c)$. The space E of the H-orbits in $G \times C$ is therefore a line bundle over $M = H\backslash G$, on which G operates to the right. Moreover, there is a function q on E taking the value $|c|^2$ on the H-orbit of any point (g, c). The space $\mathscr{H}_\chi$ can therefore be identified with the space of square-integrable sections s of E over M (square-integrable means s is measurable and $\int_M q(s)\cdot dm < \infty$). The action of G on the sections is given via the actions of G on M *and* E.

A weak form of Frobenius' reciprocity law reads as follows:
The induced representation $(\pi_\chi, \mathcal{H}_\chi)$ *is irreducible*[12] *in case the only solutions of the equation* (52) *are the constant multiples of the canonical element* u_χ.

9. **Classification of the representations of** G. Let $(\pi, \mathcal{H})$ be any irreducible representation of G. For any element ζ in the center Z of G, the operator $\pi(\zeta)$ on $\mathcal{H}$ commutes with every operator $\pi(g)$ and is therefore by the irreducibility assumption a scalar multiple of the identity. We know the characters of Z (cf. formula (42), page 369) and may conclude that there exists a unique real number λ with

$$\pi(\zeta) = \chi_\lambda(\zeta) \cdot I \qquad (\zeta \text{ in } Z). \tag{54}$$

According to von Neumann **[9]** and Stone **[11]** we have the following classification:

(a) For every $\lambda \neq 0$, there exists, up to unitary equivalence, exactly one irreducible representation $(\pi, \mathcal{H})$ satisfying (54).

(b) The case $\lambda = 0$ corresponds to the representations which are trivial on the center Z of G. They are the one-dimensional representations given by the characters ϖ_u of G:

$$\varpi_u(t, v) = e(B(v, u)) \tag{55}$$

(u is a fixed element of V).

Let us choose a normal basis $\{z, P_1, \cdots, P_n, Q_1, \cdots, Q_n\}$ of $\mathfrak{g}$. The relation (54) is equivalent to the following infinitesimal one:

$$\pi'(z) = 2\pi i\lambda \cdot I \qquad (\pi = 3.1415\cdots \text{on the right-hand side!}) \tag{56}$$

(on $\mathcal{H}_\infty$ or $\mathcal{H}_{-\infty}$ at will). The operators $\mathbf{p}_j = \pi'(P_j)$ and $\mathbf{q}_j = \pi'(Q_j)$ satisfy on $\mathcal{H}_{-\infty}$ the Heisenberg commutation relations

$$[\mathbf{p}_j, \mathbf{p}_k] = [\mathbf{q}_j, \mathbf{q}_k] = 0, \qquad [\mathbf{p}_j, \mathbf{q}_k] = 2\pi i\lambda\delta_{jk}. \tag{57}$$

10. **The Schrödinger representation of** G. Let E be any n-dimensional subspace of V with the property that B is identically zero on $E \times E$. From (40) it follows that the image $\bar{E}$ of E into G under the exponential mapping from $\mathfrak{g}$ to G is a commutative subgroup of G. The invariant subgroup $H_E = Z \cdot \bar{E}$ of G is the direct product of Z and $\bar{E}$ and there exists therefore a unique character ϖ_λ of H_E inducing χ_λ on Z and the identity on $\bar{E}$. Explicitly, one has

$$\varpi_\lambda(\iota_t \cdot e^w) = e(\lambda t) \qquad (t \in \mathbf{R}, w \in E). \tag{58}$$

[12] The representation $(\pi, \mathcal{H})$ is called irreducible in case there exists no closed vector subspace of $\mathcal{H}$, except 0 and $\mathcal{H}$ itself, invariant under every operator $\pi(g)$. A useful criterion asserts that this is the case if and only if any bounded operator in $\mathcal{H}$ commuting with every $\pi(g)$ is a scalar multiple of the identity operator.

For any $\lambda \neq 0$, we denote by $\mathcal{D}_\lambda = (\sigma_\lambda, \mathcal{H}_{\lambda, E})$ the representation of G induced by the character ϖ_λ of H_E.

To further analyse this representation, let us introduce some n-dimensional subspace E' of V, such that B induces 0 on $E' \times E'$ and that V be the direct sum of E' and E. Such a subspace E' is known to exist and to be nonunique in case $n \geqq 1$. The restriction of B to $E' \times E$ put these two vector spaces into duality. Moreover, any element of G can be uniquely written in the following form

$$g = \iota_t \cdot e^w \cdot e^{w'} \qquad (t \in \boldsymbol{R}, w \in E, w' \in E'). \tag{59}$$

By definition, $\mathcal{H}_{\lambda,E}$ consists of the functions f on G which are square-integrable modulo H_E and satisfy the relation

$$f(\iota_s \cdot e^v \cdot g) = e(\lambda s) \cdot f(g) \qquad (s \in \boldsymbol{R}, v \in E, g \in G). \tag{60}$$

It is immediate to define a Hilbert space isomorphism $f \rightleftarrows \phi$ from $\mathcal{H}_{\lambda,E}$ to $L^2(E')$ by means of the equivalent formulas[13]

$$f(g) = e(\lambda t) \cdot \phi(w'), \qquad \phi(w') = f(e^{w'}) \tag{61}$$

where g is given by (59). By means of this isomorphism, the action of G is shifted to $L^2(E')$ and is given by the following relation:

$$(\sigma_\lambda(g) \cdot \phi)(v') = e(\lambda t) \cdot e(\lambda B(v', w)) \cdot \phi(v' + w'). \tag{62}$$

Now let $\{Q_1, \cdots, Q_n\}$ be any basis of E. Since B puts E' and E into duality we can define a basis $\{P_1, \cdots, P_n\}$ and a coordinate system $\{x_1, \cdots, x_n\}$ for E' by means of the formulas

$$B(P_j, Q_k) = \delta_{jk}, \tag{63}$$

$$x_k(v') = B(v', Q_k). \tag{64}$$

According to (59), any element of G is of the form

$$\omega(t, s, u) = \iota_t \cdot e^{s_1 Q_1} \cdots e^{s_n Q_n} e^{u_1 P_1} \cdots e^{u_n P_n}$$

where t is real and $s = (s_1, \cdots, s_n)$, $u = (u_1, \cdots, u_n)$ are real n-vectors. The group law is given by

$$\omega(t, s, u) \cdot \omega(t', s', u') = \omega(t + t' + s' \cdot u, s + s', u + u') \tag{65}$$

and the operator $W(t, s, u) = \sigma_\lambda(\omega(t, s, u))$ on $L^2(E')$ is given by

$$W(t, s, u) \cdot \phi(x) = e(\lambda t) \cdot e(\lambda s \cdot x) \cdot \phi(x + u). \tag{66}$$

We are back to the Schrödinger representation with parameter $\lambda = 1/h$ (see (35) and (36)).

[13] The notation $L^2(E')$ means the space of square-integrable functions on E'.

The infinitesimal operators p_k and q_k acting on $L^2(E')$ are the differential operators

$$p_k \cdot \phi = \partial\phi/\partial x_k, \qquad q_k \cdot \phi = 2\pi i \lambda x_k \cdot \phi. \tag{67}$$

More precisely, the general theory of induced representations (see page 371) shows that the C^∞-vectors in the representation $\mathscr{D}_\lambda$ are the C^∞-functions on E' which are mapped into square-integrable functions by any finite product of the operators (67). These functions form the Schwartz' space $\mathscr{S}(E')$ whose dual is the space $\mathscr{S}'(E')$ of "tempered distributions" on E'. The action of $\mathfrak{g}$ on the space $\mathscr{S}'(E') = (\mathscr{H}_{\lambda,E'})_{-\infty}$ is still given by (67).

The irreducibility of Schrödinger representation is a familiar result, but it is instructive to derive it from our general irreducibility criterion (see page 372) and the following elementary lemma in distribution theory.

LEMMA 1. *Any distribution T on the real n-space E' satisfying the conditions*

$$x_k \cdot T = 0 \qquad (k = 1, \cdots, n) \tag{68}$$

is a constant multiple of the Dirac distribution δ defined by $\delta(\phi) = \phi(0)$ for any test-function ϕ.

Using the classification of the irreducible representations of G given on page 372 and using the preceding result, we obtain easily the following result.

THEOREM 1. *Let $\{\varpi, \mathscr{H}\}$ be any irreducible representation of G, nontrivial on the center Z of G, and let E be any n-dimensional subspace of V on which B induces the zero form. The set of solutions of the equation*

$$\varpi'(X) \cdot v = 0 \qquad \textit{for every } X \textit{ in } E \tag{69}$$

is a one-dimensional subspace of $\mathscr{H}_{-\infty}$.

11. **Some discrete subgroups.** Let L be a lattice[14] in V such that B take integral values on $L \times L$; the complementary lattice L' is the set of all vectors v in V such that $B(v, \lambda)$ be an integer for every λ in L; it obviously contains L. The set of elements of the form $\iota_t \cdot e^\lambda$ with t real and λ in L is an invariant subgroup Γ_L of G; the subgroup $\Gamma_{L'}$ is defined in a similar way. Let us consider also the discrete subgroup Δ of the center Z of G consisting of the elements ι_m with m an integer. The group $\Gamma_{L'}$ is nothing else than the set of all g's for which the commutator $g\gamma g^{-1}\gamma^{-1}$ lies in Δ for every γ in Γ_L.

We denote by Ξ the group of all characters of Γ_L taking the value 1 on all of Δ; we have $\Xi = \bigcup_m \Xi_m$ (disjoint union) where Ξ_m is the set of characters of Γ_L extending the character χ_m of Z (m runs over the set of integers). The general form of the elements in Ξ_m is given as follows

$$\Psi_{m,F}(\iota_t \cdot e^\lambda) = e(mt) \cdot e(\tfrac{1}{2}F(\lambda)), \tag{70}$$

[14] That is, a discrete subgroup of V generating it as a vector space, or equivalently, the set of vectors with integral coordinates in a suitable basis of V.

where F is any real-valued function on L, defined modulo 2, satisfying the congruence:

(71) $$F(\lambda + \mu) \equiv F(\lambda) + F(\mu) + m \cdot B(\lambda, \mu) \qquad (\text{mod } 2).$$

More simply, the characters in Ξ_0 are given by

(72) $$\Psi(\iota_t \cdot e^\lambda) = e(B(v_0, \lambda))$$

where v_0 is a fixed element of V, defined modulo L' by this relation.

Let m be nonzero and let $\Psi_{m,F}$ and $\Psi_{m,F'}$ be two elements of Ξ_m. We can write $\Psi_{m,F'} = \Psi \cdot \Psi_{m,F}$ with some Ψ in Ξ_0. Using formula (72), we get after easy manipulations

(73) $$\Psi_{m,F'}(\gamma) = \Psi_{m,F}(g \cdot \gamma \cdot g^{-1}) \qquad (\gamma \text{ in } \Gamma_L)$$

where we can take for g the element $g_0 = \exp(m^{-1} \cdot v_0)$ of G; the elements g qualifying for (73) form the whole coset $g_0 \cdot \Gamma_{m^{-1}L'}$.

A more explicit description of the situation can be given as follows. According to elementary divisor theory, there exists a normal basis $\{z, P_1, \cdots, P_n, Q_1, \cdots, Q_n\}$ of $\mathfrak{g}$ and integers $e_1, \cdots, e_n$ such that the elements of L (resp. L') are the vectors

(74) $$\lambda = t_1 \cdot P_1 + \cdots + t_n \cdot P_n + s_1 \cdot Q_1 + \cdots + s_n \cdot Q_n$$

whose coordinates are solutions of the congruences

(75) $$s_j \equiv 0, \qquad e_j^{-1} \cdot t_j \equiv 0 \ (\text{mod } 1) \qquad \text{for } j = 1, \cdots, n,$$

(resp.

(75′) $$e_j \cdot s_j \equiv 0, \qquad t_j \equiv 0 \ (\text{mod } 1) \qquad \text{for } j = 1, \cdots, n).$$

As a corollary, we get that the index $[L' : L]$ is the square of the integer $e = e_1 \cdots e_n$.

A special instance of a solution of the functional Equation (71) is given as follows

(76) $$F_0(\lambda) = m \cdot (t_1 s_1 + \cdots + t_n s_n).$$

The general solution is given by

(77) $$F(\lambda) \equiv F_0(\lambda) + a_1 s_1 + \cdots + a_n s_n + e_1^{-1} b_1 t_1 + \cdots + e_n^{-1} b_n t_n \qquad (\text{mod } 2)$$

where $a_1, \cdots, a_n, b_1, \cdots, b_n$ are real numbers defined modulo 1. Let us remark that for m even, we might as well take $F_0 = 0$ as a particular solution of the congruence (71).

A particularly important special case is provided by the so-called "principal lattices," that is the lattices L equal to their complementary L'. For such an L, the commutator group of Γ_L is equal to Δ, and Ξ is therefore the set of all characters of Γ_L; moreover any two characters belonging to the same Ξ_m (with $m \neq 0$) are conjugate to each other by some element of G well-defined modulo $\Gamma_{m^{-1}L}$. Finally in case of a principal lattice, the "elementary divisors" $e_1, \cdots, e_n$ are all equal to 1.

Assume L to be principal. The Equation (71) is then satisfied for at least one integral-valued F; in case m is even, it suffices to take $F = 0$. Assuming therefore m to be odd, denote by $\bar{L}$ the vector space $L/2L$ over the field with two elements By reduction modulo 2, the form B defines a symmetric bilinear form $\bar{B}$ on $\bar{L} \times \bar{L}$ and the integral-valued solutions of (71) correspond via reduction modulo 2 to the quadratic forms $\bar{F}$ on $\bar{L}$ whose associated bilinear form is $\bar{B}$. These quadratic forms fall into two equivalence classes according to the value of their "Arf invariant." Using again a normal basis $\{z, P_1, \cdots, P_n, Q_1, \cdots, Q_n\}$ for which L is the set of vectors with integral coordinates in V, we get the following reduced forms for the F falling in either one of the two classes:

$$F'(\lambda) = t_1 s_1 + \cdots + t_n s_n, \tag{78}$$

$$F''(\lambda) = t_1 s_1 + \cdots + t_n s_n + s_1^2 + t_1^2. \tag{79}$$

12. **The lattice representations.** We proceed now to describe a class of representations of G which have so far played no role in quantum mechanics.

We fix a lattice L such that B takes integral values on $L \times L$, an integer $m \neq 0$ and a function F solution of (71). The representation of G induced by the character $\Psi_{m,F}$ of Γ_L shall be denoted $\mathscr{D}_{L,m,F}$. Using the correspondence $f \rightleftarrows \phi$ expressed by the equivalent relations

$$\phi(v) = f(e^v), \qquad f(\iota_t \cdot e^v) = e(mt) \cdot \phi(v), \tag{80}$$

we shift the action of G to the space $\mathscr{H}_{L,m,F}$ of functions ϕ on V subjected to the following restrictions:

(a) The function ϕ is Borel-measurable on V.

(b) The integral $\int_P |\phi(v)|^2 \, dv$ is finite for every fundamental domain P of L acting by translation on V (for instance a suitable parallelotope).

(c) Functional equation:

$$\phi(v + \lambda) = e\left(\frac{1}{2}F(\lambda) + \frac{m}{2}B(v, \lambda)\right) \cdot \phi(v) \tag{81}$$

for v in V and λ in L.

The action of G in $\mathscr{H}_{L,m,F}$ is given as follows

$$(U_{v'}\phi)(v) = \phi(v + v') \cdot e\left(\frac{m}{2}B(v, v')\right) \tag{82}$$

where $U_{v'} = \pi_m(e^{v'})$ is the operator corresponding to the element $e^{v'}$ of G. In what follows, we consider only the case where $m = 1$, the general case going easily over to that case by replacing B by $m \cdot B$ throughout. We shall omit the index 1 in the notations $\Psi_{1,F}$, $\mathscr{H}_{L,1,F}$, $\mathscr{D}_{L,1,F}$ and π_1.

We give now an analysis of irreducibility for the representation $\mathscr{D}_{L,F}$; we shall eventually prove that *the irreducibility is at hand if and only if L is a principal lattice.* To every λ' in L', we can associate an operator $A_{\lambda'}$ commuting to $\pi(G)$

and given via left translation

$$(83) \qquad (A_{\lambda'} f)(g) = f(e^{\lambda'} \cdot g)$$

or equivalently (see formula (80)) by

$$(84) \qquad (A_{\lambda'}\phi)(v) = e(\tfrac{1}{2}B(\lambda', v))\phi(v + \lambda')$$

for any function ϕ in $\mathscr{H}_{L,F}$. Using (40), we get

$$(85) \qquad A_{\lambda'} \cdot A_{\mu'} = e(\tfrac{1}{2}B(\lambda', \mu')) \cdot A_{\lambda' + \mu'}$$

while (81) takes the form

$$(86) \qquad A_\lambda = e(\tfrac{1}{2}F(\lambda)) \cdot I \qquad (\lambda \text{ in } L).$$

LEMMA 2. *Let S be any set of representatives for the cosets of L' modulo L. The operators A_s for s in S form a basis of the algebra of all operators in $\mathscr{H}_{L,F}$ commuting to $\pi(G)$.*

The proof runs as follows. First of all, the infinitesimal representation is given by

$$(87) \qquad (\pi'(X) \cdot \phi)(v) = \theta_X\phi(v) + \pi i \cdot B(v, X) \cdot \phi(v) \qquad (X \text{ in } V)$$

where θ_X is the Lie derivative[15] associated to the constant vector field on V with value X. More precisely, $(\mathscr{H}_{L,F})_\infty$ is the set of C^∞-solutions of the functional Equations (81) such that $\pi'(X_1) \cdots \pi'(X_p) \cdot \phi$ be square integrable modulo L for every sequence of elements $X_1, \cdots, X_p$ in V, and $(\mathscr{H}_{L,F})_{-\infty}$ is the set of distributions on V which can be expressed as finite sums of derivatives $\pi'(X_1) \cdots \pi'(X_p) \cdot \phi$ of functions ϕ belonging to $\mathscr{H}_{L,F}$. These distributions satisfy the functional Equation (81) in a symbolic sense. The canonical element (see page 371) expresses the distribution u given on any test function ϕ by[16]

$$(88) \qquad u(\phi) = \sum_{\lambda \in L} e(\tfrac{1}{2}F(\lambda)) \cdot \phi(\lambda).$$

The action of $A_{\lambda'}$ on the distributions belonging to $(\mathscr{H}_{L,F})_{-\infty}$ be expressed by the same formula which works foı functions, at least when suitably interpreted in a symbolic way. This entails the following formula

$$(89) \qquad (A_{\lambda'}u)(\phi) = \sum_{\lambda \in L} e(\tfrac{1}{2}F(\lambda)) \cdot e(\tfrac{1}{2}B(\lambda', \lambda)) \cdot \phi(\lambda - \lambda')$$

for $A_{\lambda'}u$.

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[15] Defined by

$$\theta_X f(v) = \lim_{t \to 0} \frac{1}{t}[f(v + tX) - f(v)].$$

[16] In this case $F = 0$, this distribution deserves to be called *Poisson distribution* because of its significance for the Poisson summation formula.

According to the general theory of representations, any operator A in $\mathscr{H}_{L,F}$ commuting to $\pi(G)$ has a natural extension to $(\mathscr{H}_{L,F})_{-\infty}$ which commutes with the action of G on $(\mathscr{H}_{L,F})_{-\infty}$. If $t = A \cdot u$, we have therefore

(90) $$\pi(e^{\lambda}) \cdot t = e(\tfrac{1}{2}F(\lambda)) \cdot t$$

for every λ in L. Moreover, $A \cdot u$ is 0 if and only if A is 0. It remains therefore to prove that any distribution in $(\mathscr{H}_{L,F})_{-\infty}$ solution of (90) is a linear combination of the distributions $A_s \cdot u$, which is tantamount proving the following lemma.

LEMMA 3. *Any distribution t on V satisfying the symbolic equations*

(91) $$t(v + \lambda) = e(\tfrac{1}{2}F(\lambda)) \cdot e(\tfrac{1}{2}B(v, \lambda)) \cdot t(v),$$

(92) $$e(\tfrac{1}{2}B(v, \lambda)) \cdot t(v + \lambda) = e(\tfrac{1}{2}F(\lambda)) \cdot t(v)$$

for every λ in L, is of the form $t = \sum_{s \in S} T(-s) \cdot A_s u$ with suitable constants $T(-s)$.

We can replace the system of equations (91) and (92) by the equivalent system consisting of (91) and

(92′) $$t(v) = e(B(v, \lambda)) \cdot t(v).$$

Since L' is by definition the set of common zeros of the functions $e(B(v, \lambda)) - 1$ for λ in L, an easy transversality argument shows that any solution of (92′) is given by[17]

(93) $$t(v) = \sum_{\lambda' \in L'} T(\lambda') \cdot \delta(v - \lambda')$$

with a suitable complex-valued function T on L'. This being so, Equation (91) amounts to the relation (for λ in L and λ' in L')

(94) $$T(\lambda' + \lambda) = e(\tfrac{1}{2}F(\lambda)) \cdot (-1)^{B(\lambda', \lambda)} \cdot T(\lambda')$$

and implies therefore

$$t = \sum_{s \in S} T(-s) \sum_{\lambda \in L} e(\tfrac{1}{2}F(\lambda)) \cdot (-1)^{B(s, \lambda)} \cdot \delta(v - \lambda + s)$$

that is

$$t = \sum_{s \in S} T(-s) \cdot A_s u.$$

We can now state the main result of this section.

THEOREM 2. *Let L be any lattice in V such that B takes integral values on $L \times L$, and let F be any solution of the Equation* (71) *with $m = 1$. Let L' be the lattice complementary to L and put $[L' : L] = e^2$.*[18] *Finally, let $(\varpi, \mathscr{H})$ be any irreducible representation of G such that $\varpi(\iota_t) = e(t) \cdot I$ for every real t.*

[17] By definition, the Dirac distribution $\delta(v - a)$ takes the value $\phi(a)$ on any test-function ϕ. For instance, (88) can be written $u(v) = \sum_{\lambda \in L} e(\frac{1}{2}F(\lambda)) \cdot \delta(v - \lambda)$ and similarly for (89).

[18] The index $[L' : L]$ is equal to the determinant of the matrix $\{B(v_i, v_j)\}$ where $\{v_1, \cdots, v_{2n}\}$ is any basis of V for which L is the set of vectors with integral coordinates.

(a) *The induced representation* $\mathscr{D}_{L,F}$ *is isomorphic to the direct sum of e copies of* $(\varpi, \mathscr{H})$.

(b) *The set of solutions of the equations*

$$\varpi(e^{\lambda}) \cdot t = e(\tfrac{1}{2}F(\lambda)) \cdot t \qquad (\lambda \text{ in } L) \tag{95}$$

is an e-dimensional subspace of $\mathscr{H}_{-\infty}$.

Since the algebra of operators in $\mathscr{H}_{L,F}$ commuting to $\pi(G)$ is finite-dimensional (its dimension being in fact equal to e^2), the representation $\mathscr{D}_{L,F}$ splits into a direct sum of finitely many irreducible components $\mathscr{D}_1, \cdots, \mathscr{D}_p$. Since $\pi(\iota_t) = e(t) \cdot I$, the classification of the representations of G shows these representations are indeed equivalent to $(\varpi, \mathscr{H})$. The commuting algebra is therefore isomorphic to the algebra of all $p \times p$ matrices and a dimension argument gives $p = e$. This proves (a). As to (b), it suffices to use (a) and to remark that the set of solutions of Equation (90) is an e^2-dimensional subspace of $(\mathscr{H}_{L,F})_{-\infty}$ by Lemma 3.

13. **Fock representation.** In order to define invariantly the Fock representation, we need a real number $\lambda \neq 0$ and an operator J in V with the properties:

$$J^2 v = -v, \tag{96}$$

$$B(Jv, Jv') = B(v, v'), \tag{97}$$

$$B(v, Jv) \geqq 0, \tag{98}$$

for any pair v, v' of elements of V. We have also to consider the complexification V_c of V, that is a complex vector space containing V such that every one of its elements can be written uniquely as $x = v + iv'$ with v and v' in V. The conjugate $\bar{x}$ of the vector x is by definition $v - iv'$. The bilinear form B on $V \times V$ extends to a complex bilinear form B_c on $V_c \times V_c$. The complex extension J_c of J to V_c has a square equal to minus the identity operator; it has therefore the eigenvalues i and $-i$ with respective eigenspaces some subspace W of V_c and its conjugate $\overline{W}$ (set of all vectors $\bar{x}$ for x in W). Using (97), one sees that B_c induces the zero form on both W and $\overline{W}$.

If we replace in the definition of G the real pairs (t, v) by complex ones (that is t is a complex number and v is in V_c) and still use the rule (37) to compute the product, we define a Lie group G_c, containing G as a closed subgroup, and with Lie algebra the complexification $\mathfrak{g}_c$ of $\mathfrak{g}$. Moreover the set of pairs $(t, \bar{x})$ with t complex and x in W is a closed subgroup P of G_c such that $G \cap P = Z$ and $G \cdot P = G_c$. We define a continuous homomorphism δ_λ from P to the multiplicative group of nonzero complex numbers by

$$\delta_\lambda(t, \bar{x}) = e(\lambda t). \tag{99}$$

With all these conventions in mind, we can define the Fock representation as a kind of *holomorphic induced representation.*[19] Indeed, it acts on a Hilbert space consisting of all functions f on G_c subjected to the following restrictions:

(a) f is holomorphic;

(b) one has $f(pg) = \delta_\lambda(p) \cdot f(g)$ for p in P and g in G_c;

(c) the integral $\int_{Z \backslash G} |f(g)|^2 \, dg$ is finite.[20]

The scalar product is given by the integral

$$(f|f') = \int_{Z \backslash G} \overline{f(g)} \cdot f'(g) \, dg \tag{100}$$

and the group G acts by the right translations defined by

$$(R_h f)(g) = f(gh). \tag{101}$$

In the applications, it is more convenient to shift everything to V as follows. We denote by V_J the complex vector space having V as underlying real space in which J is the scalar multiplication by i. On V_J, there is a unique hermitian form H having B as imaginary part; explicitly, one has:

$$H(v, v') = B(v, Jv') + i \cdot B(v, v') \tag{102}$$

and, according to (98), one has $H(v, v) \geqq 0$ for any v.

The correspondence $f \rightleftarrows \phi$ devised by the formula

$$\phi(v) = e^{\pi\lambda H(v,v)/2} \cdot f(e^v) \tag{103}$$

maps isomorphically the space of the Fock representation onto the Hilbert space $\mathscr{F}_J$ whose elements are the C^∞-functions ϕ on V satisfying the properties

$$\theta_{JX}\phi = i \cdot \theta_X \phi \qquad \text{(for every } X \text{ in } V), \tag{104}$$

$$\int_V e^{-\pi\lambda H(v,v)} |\phi(v)|^2 \, dv < \infty. \tag{105}$$

The equation (104) is nothing else than the set of Cauchy-Riemann equations in an invariant guise and expresses that ϕ is holomorphic on V_J. As to the scalar product, it is given by

$$(\phi|\phi') = \int_V e^{-\pi\lambda H(v,v)} \overline{\phi(v)} \phi'(v) \, dv \tag{106}$$

and the operator associated to $\iota_t \cdot e^v$ is $\varpi_J(\iota_t \cdot e^v) = e(\lambda t) \cdot U_v$ where U_v is expressed as follows

$$(U_v \phi)(v') = e^{-\pi\lambda[H(v,v)/2 + H(v,v')]} \cdot \phi(v + v'). \tag{107}$$

[19] I thank heartfully J. Dixmier for having pointed out to me the importance of this notion and its bearing to our problems.

[20] By condition (b) for $p = \iota_t$ we get that $|f|^2$ is constant on every coset Zg, giving a meaning to the previous integral.

The infinitesimal representation ϖ'_J associated to ϖ_J is given by

$$\varpi'_J(X)\cdot\phi = \theta_X\cdot\phi - \pi\lambda H_X\cdot\phi \tag{108}$$

where H_X is the linear function $v \mapsto H(X, v)$ on V. We have to make the usual proviso, that is $(\mathscr{F}_J)_\infty$ is the set of all holomorphic functions ϕ on V_J such that $\varpi'_J(X_1)\cdots\varpi'_J(X_p)\cdot\phi$ is in $\mathscr{F}_J$ whatever $X_1, \cdots, X_p$ is in V, and $(\mathscr{F}_J)_{-\infty}$ consists of the finite sums of functions of the form $\varpi'_J(X_1)\cdots\varpi'_J(X_p)\cdot\phi$ with $X_1, \cdots, X_p$ in V and ϕ in $\mathscr{F}_J$. Taking into account the Cauchy-Riemann equations (104) and the obvious relation $H_{JX} = -iH_X$, we can transform (108) as follows[21]

$$\varpi'_J(Y)\cdot\phi = \theta_X\phi, \tag{109}$$

$$\varpi'_J(\bar{Y})\cdot\phi = -\pi\lambda H_X\cdot\phi \tag{110}$$

where Y is the unique element in W such that $X = Y + \bar{Y}$, that is

$$Y = \tfrac{1}{2}(X - i\cdot JX). \tag{111}$$

The main result concerning the Fock representation can be stated as follows.

THEOREM 3. *Let J be any operator in V satisfying the relations* (96) *to* (98) *and $\lambda \neq 0$ be real. Let W be the subspace of the complexification V_c of V associated to the eigenvalue i of the complex extension J_c of J to V_c.*

(a) *The Fock representation $(\varpi_J, \mathscr{F}_J)$ is irreducible.*

(b) *If $(\varpi, \mathscr{H})$ is any irreducible representation of G which is nontrivial on the center Z of G, the vectors in $\mathscr{H}_{-\infty}$ annihilated by $\varpi'(W)$ form a one-dimensional subspace of $\mathscr{H}_\infty$.*

We first prove (b) in case of the Fock representation. According to the description of $(\mathscr{F}_J)_{-\infty}$ and formula (109), an element of $(\mathscr{F}_J)_{-\infty}$ annihilated by $\varpi'_J(W)$ is a holomorphic function ϕ on V_J such that $\theta_X\phi = 0$ for every X in V, that is a constant.

For every real t, one has $\varpi_J(\iota_t) = e(\lambda t)\cdot I$. We may assume $\varpi(\iota_t) = e(\lambda t)\cdot I$ in view of the arbitrariness of λ. According to von Neumann results **[9]**, the Fock representation is therefore isomorphic to the direct sum of a certain number m (finite or not) of copies of $(\varpi, \mathscr{H})$. Accordingly, the subspace T of $(\mathscr{F}_J)_{-\infty}$ annihilated by $\varpi'_J(W)$ contains the (algebraic) direct sum of m copies of the space S in $\mathscr{H}_{-\infty}$ annihilated by $\varpi'(W)$. Since T is one-dimensional, we get $m = 1$ and $\dim S = 1$. This proves assertions (a) and (b) in Theorem 3.

We conclude by some explicit formulas. Since H is a positive nondegenerate hermitian form on V_J we can choose a (complex) basis $\{P_1, \cdots, P_n\}$ for V_J such that $H(P_k, P_l) = \delta_{kl}$ and set $Q_j = J\cdot P_j$. It is easy to see that

$$\{z, P_1, \cdots, P_n, Q_1, \cdots, Q_n\}$$

[21] We have extended in the obvious way ϖ'_J to a representation of the complex Lie algebra $\mathfrak{g}_c$.

is a normal basis of $\mathfrak{g}$. Moreover, if we denote by $z_1, \cdots, z_n$ the complex-linear functions on V_J defined by $z_k(P_l) = \delta_{kl}$, the monomials

$$M_\alpha = \lambda^{n/2} \prod_{j=1}^{n} \frac{(\pi\lambda)^{\alpha_j/2}}{(\alpha_j!)^{1/2}} z_j^{\alpha_j} \qquad \alpha = (\alpha_1, \cdots, \alpha_n) \tag{112}$$

form an orthonormal basis of $\mathscr{F}_J$.[22] According to (109) and (110), the infinitesimal operator $\varpi'_J(P_j - iQ_j)$ is twice the derivation with respect to the complex variable z_j and $\varpi'_J(P_j + iQ_j)$ is multiplication by $-2\pi\lambda z_j$.

14. **Definition of theta functions.** The whole machinery of Riemann forms can now be set up. To summarize, let be given:

— a real vector space V of finite dimension $2n$;
— a nondegenerate alternating bilinear form B on $V \times V$;
— an operator J on V satisfying to the relations (96) to (98);
— a lattice L in V such that B takes integral values on $L \times L$;
— a real-valued function F on V such that

$$F(\lambda + \mu) \equiv F(\lambda) + F(\mu) + B(\lambda, \mu) \qquad (\text{mod } 2) \tag{113}$$

for any pair λ, μ of elements of L.

By means of these data, a Fock representation $(\varpi_J, \mathscr{F}_J)$ (with $\lambda = 1$) is defined whose irreducibility follows from Theorem 3. By Theorem 2, the solutions of the equation

$$\varpi_J(e^\lambda) \cdot t = e(\tfrac{1}{2}F(\lambda)) \cdot t \qquad (\text{for every } \lambda \text{ in } L) \tag{114}$$

form an e-dimensional subspace Θ of $(\mathscr{F}_J)_{-\infty}$. Made explicit, the previous equation reads as follows

$$t(v) = t(v + \lambda) \cdot \exp -\pi[\tfrac{1}{2}H(\lambda, \lambda) + H(\lambda, v) + i \cdot F(\lambda)] \tag{115}$$

and is nothing else than the well-known functional equation defining the theta functions. We get Frobenius' theorem that the dimension of the space of solutions of (115) is given as the square root of the discriminant of B with respect to L.

A few questions to conclude: The group G is nothing but a special instance of a real nilpotent algebraic group. How can one extend to the general case the three methods given here to generate irreducible representations of such a group? What kind of functions on such a group play the role of theta functions?

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[22] Following Bergman's well-known procedure, we ought to introduce the kernel

$$K(v, v') = \sum_\alpha \overline{M_\alpha(v)} \cdot M_\alpha(v')$$

given here by $K(v, v') = e^{\pi\lambda H(v,v')}$. Its intrinsic meaning is as follows. For every v in V, the function $v' \mapsto K(v, v')$ is an element K_v of $\mathscr{F}_J$ and we have $(K_v|f) = f(v)$ for every function f in $\mathscr{F}_J$.

References

1. F. Bruhat, *Sur les représentations induites des groupes de Lie*, Bull. Soc. Math. France **84** (1956), 97–205.

2. P. Cartier, *Analyse spectrale et théorème de prédiction statistique de Wiener*, Séminaire Bourbaki, 13^e année, 1960/61, exp. 218.

3. J. Dixmier, *Sur la relation* $i(PQ - QP) = I$, Compositio Math. **13** (1958), 263–269.

4. L. Gårding, *A note on continuous representations of Lie groups*, Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 331–332.

5. I. Gelfand and I. Piateskii-Shapiro, *Theory of representations and theory of automorphic functions*, Amer. Math. Soc. Transl. (2) **26** (1963), 173–200.

6. T. Kato, *On the commutation relation* $AB - BA = c$, Arch. Rational Mech. Anal. **10** (1962), 273–275.

7. G. Mackey, *Induced representations of locally compact groups* I., Ann. of Math. **55** (1952), 101–139; II. **58** (1953), 193–221.

8. E. Nelson, *Analytic vectors*, Ann. of Math. **70** (1959), 572–615.

9. J. von Neumann, *Die Eindeutigkeit der Schrödingerschen Operatoren*, Math. Ann. **104** (1931), 570–578.

10. F. Rellich, *Der Eindeutigkeitssatz für die Lösungen der quantenmechanischen Vertauschungsrelationen*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. (1946), 107–115.

11. M. Stone, *Linear transformations in Hilbert space*, III. *Operational methods and group theory*, Proc. Nat. Acad. Sci. U.S.A. **16** (1930), 172–175.

12. A. Weil, *Sur certains groupes d'opérateurs unitaires*, Acta Math. **113** (1964), 143–211.

13. H. Weyl, *The theory of groups and quantum mechanics*, Dover Publications, New York, 1950.