The set of prime numbers of the form $6n + 1$, is infinite, and so is the twin primes set (Denise Chemla, $5^{th} of june, 2012$)

1 Recalls

Let us recall Pascal Triangle content, made of binomial coefficients.

								1									$\rightarrow$	1
							1		1								$\rightarrow$	2
						1		2		1							$\rightarrow$	4
					1		3		3		1						$\rightarrow$	8
				1		4		6		4		1					$\rightarrow$	16
			1		5		10		10		5		1				$\rightarrow$	32
		1		6		15		20		15		6		1			$\rightarrow$	64
	1		7		21		35		35		21		7		1		$\rightarrow$	128
1		8		28		56		70		56		28		8		1	$\rightarrow$	256

The sum of all coefficients from the line n is equal to : 2^n.

$$\sum_{p=1}^{n} C_n^p = 2^n$$

Let us recall the number of ways to have one or more zero among 4 numbers. It is equal to $\sum_{p=1}^{n} C_n^p - 1$:

0	−	−	−	1
−	0	−	−	2
−	−	0	−	3
−	−	−	0	4
0	0	−	−	1
0	−	0	−	2
0	−	−	0	3
−	0	0	−	4
−	0	−	0	5
−	−	0	0	6
0	0	0	−	1
0	0	−	0	2
0	−	0	0	3
−	0	0	0	4
0	0	0	0	1

Finally, let us recall the modular residues writings of first natural numbers (*modulo prime numbers*). To obtain those writings, we add the t-uple $(1, 1, 1, 1, \ldots)$ successively to each residues word.

mod	2	3	5	7	11	13	...
1	1	1	1	1	1	1	...
2	0	2	2	2	2	2	...
3	1	0	3	3	3	3	...
4	0	1	4	4	4	4	...
5	1	2	0	5	5	5	...
6	0	0	1	6	6	6	...
7	1	1	2	0	7	7	...
8	0	2	3	1	8	8	...
9	1	0	4	2	9	9	...
10	0	1	0	3	10	10	...
11	1	2	1	4	0	11	...
12	0	0	2	5	1	12	...
13	1	1	3	6	2	0	...

2 The set of prime numbers of the form $6n + 1$ is infinite

We demonstrate by recurrence the property $P(p_i)$ that is that there is always one prime number of the form $6n + 1$ that is greater than $\#p_{i-1}$ and lesser than $\#p_i$. We recall that the notation $\#p_i$ is used to designate the product of all prime numbers that are between 2 and p_i.

1) First step of the recurrence : the property is true for $p_3 = 5$. There is a prime number 7 of the form $6n + 1$ that is between $\#3 = 2.3 = 6$ and $\#5 = 2.3.5 = 30$.

2) Recurrence from $P(p_i)$ to $P(p_{i+1})$: the property is true for p_i signifies that there is a prime number of the form $6n + 1$ between $\#p_{i-1}$ and $\#p_i$. Let us show that there is another prime number of the form $6n+1$ that is between $\#p_i$ and $\#p_{i+1}$. Between $\#p_i$ and $\#p_{i+1}$, there are $\#p_i(p_{i+1}-1)$ different numbers. But among those numbers, only $2^{i+1} - 1$ contain at least one zero in their modular residues writing. As $\#p_i(p_{i+1} - 1)$ is very greater than 2^{i+1} (cf the inequality just after for $p_i = 7$), the numbers that have no zero in their writing are prime. Only $\frac{1}{6}$ of such numbers are of the form $6n - 1$. They can't all be of the form $6n - 1$. One of the numbers that is not of the form $6n - 1$ must be of the form $6n + 1$. It hasn't yet been found until now because numbers between $\#p_i$ and $\#p_{i+1}$ are all different from those encountered until $\#p_i$.

If $p_i = 7$, we have the following inequality :

$$\#p_i(p_{i+1} - 1) \gg 2^{i+1} - 1$$
$$2.3.5.7.(11 - 1) \gg 2.2.2.2.2 - 1$$

3 The set of twin primes is infinite

As we can see and understand it in the table that provides the modular residues writings of first natural numbers, there are as many as 2-uples of twin primes than even numbers "just between" 2 twin prime numbers. Let us observe the modular residues writings of such even numbers. An even natural number $p_i + 1$ that is between two twin prime numbers p_i and $p_i + 2$ must be :

- congruent to 1 (mod $\mathrm{p_i}$) ;
- not congruent to $\mathrm{p_k} - 1$ and not congruent to 1 for every $\mathrm{k} < \mathrm{i}$.

To demonstrate by recurrence that there always exists such an even natural number different from those yet encountered between $\#p_{i-1}$ and $\#p_i$, we have to demonstrate that the number of even natural numbers "just between"' two twin prime numbers (that we will call *NbEvenMiddleTwins*) is always strictly greater than 1. The newness of numbers that are between $\#p_i$ and $\#p_{i+1}$ guarantees that we had found a new even natural number "just between" two new twin prime natural numbers and so that the set of twin prime natural numbers is infinite.

$$NbEvenMiddleTwins = \frac{1}{p_i}\left[\#p_i(p_{i+1} - 1) - \left(\sum_{k=1}^{i-1} 2^k.C_{i-1}^k - 1\right)\right]$$

We have to multiply by $\frac{1}{p_i}$ to calculate the number of words that contain a 1 as last letter (i.e. $(mod\, p_i)$).

The expression $\#p_i(p_{i+1}-1)$ counts the number of words between $\#p_i$ and $\#p_{i+1}$.

We substract from this total number of words the value of $\sum_{k=1}^{i-1} 2^k.C_{i-1}^k - 1$ that corresponds to the number of words that contain at least one $p_k - 1$ or one 1 as a letter among the $i-1$ first letters from the modular residues writing (i.e. $(mod\ p_k)$, for all p_k strictly lesser than i). The residue 1 corresponds to the fact that the natural number p_i just before the even number $p_i + 1$ is not prime although the residue $p_k - 1$ corresponds to the fact that the number $p_i + 2$ just after the even number $p_i + 1$ is not prime.

The number of words to be eliminated being equal to 3^{i-1} is, as in previous section, very lesser that the total number of words and we are so ensured to find between to successive primorials an even number that is "just between" two twin prime numbers.

4 Existence of a Goldbach decomponent for every even natural number strictly greater than 4

Goldbach Conjecture (7 juin 1742) asserts that every even natural number strictly greater than 4 is the sum of two odd prime natural numbers. If we note $\mathbb{P}^*$ the set of odd prime natural numbers :

$$\mathbb{P}^* = \{p_1 = 3, p_2 = 5, p_3 = 7, p_4 = 11, \ldots\},$$

we can write Goldbach Conjecture as follows :

$$\forall\, n \in 2\mathbb{N}\backslash\{0,2,4\},\ \exists\, p \in \mathbb{P}^*,\ p \leq n/2,\ \exists\, q \in \mathbb{P}^*,\ q \geq n/2,\ \ n = p + q.$$

In the following, n being given, we note : $\mathbb{P}^*(n) = \{x \in \mathbb{P}^*/x \leq n\}$,

An odd prime natural number that is never congruent to n, a given even natural number strictly greater than 4, according to no module belonging to $\mathbb{P}^*(n)$, is a Goldbach decomponent of n.
Indeed,

$$\begin{aligned} \forall\, n \in 2\mathbb{N}\backslash\{0,2,4\},\ \exists\, p \in \mathbb{P}^*(n),\ \forall\, m \in \mathbb{P}^*(n), \quad & p \not\equiv n\ (mod\ m) \\ & \Leftrightarrow n - p \not\equiv 0\ (mod\ m) \\ & \Leftrightarrow n - p\ is\ prime. \end{aligned}$$

We are going to look for n, an even natural number that is between two successive primorials $\#p_i$ and $\#p_{i+1}$ an odd prime natural number that is lesser than $\#p_i$ and that is not congruent to n according to every odd prime module lesser than p_i. We are going to show that among the natural numbers that are lesser than $\#p_i$, there is at least one natural number that is neither congruent to 0 (so it is prime) nor congruent to n (so it shares none of its residues with n). We saw in precedent section that 3^i natural numbers lesser than $\#p_i$ have at least one of their residues that is null or equal to a given value (in our case the n's residue according to a considerated module). But as 3^i is always very lesser than $\#p_i$, there always exists a natural number lesser than $\#p_i$ that is neither congruent to 0 according to none module nor congruent to n according to none module lesser than p_i. This prime number is a Goldbach decomponent of n.

$$Si \quad \#p_i < n < \#p_{i+1} \quad alors \quad NbDecompGoldbach(n) > \#p_i - 3^i > 1.$$