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The Maslov Index on the Simply Connected Covering Group and the Metaplectic Representation

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In this paper we define a half-integer valued function on the simply connected covering group of the symplectic group, related to the Maslov index of curves on the Lagrangian Grassmannian and use it to write down explicitly the operators of the metaplectic or Oscillator representation. We also elucidate the relationship of the so-called Maslov bundle (as defined by Hormander) to the metaplectic representation.

1. INTRODUCTION

In this paper we go back and look at the construction of the metaplectic (or oscillator) representation. The metaplectic representation was discovered by Segal, Shale [S], and Weil [W] in the early sixties and is based on two observations. One, the Stone–von Neumann theorem which asserts that for the Heisenberg group, given a central character χ, there exists up to unitary equivalence a unique irreducible unitary representation v with χ as it central character. Second, the symplectic group, Sp_n, is the group of continuous automorphisms of the Heisenberg group which fix the center pointwise. Thus the two representations v and $v \circ \sigma$ are equivalent or for each $\sigma \in Sp_n$, there exists a unitary operator $r(\sigma)$ such that $r(\sigma)\, vr(\sigma)^{-1} = v \circ \sigma$. Irreducibility of v implies that $r(\sigma)$ is uniquely determined up to a phase factor. If you make a choice of the operators $\sigma \to r(\sigma)$, one gets only a projective representation, i.e., $r(\sigma_1 \sigma_2) = c(\sigma_1, \sigma_2)\, r(\sigma_1)\, r(\sigma_2)$, for suitable scalars $c(\sigma_1, \sigma_2)$. It is known that this 2-cocycle $c(\sigma_1, \sigma_2)$ is ± 1 valued for a suitable choice $\sigma \to r(\sigma)$. Equivalently (see [W], or [B–W]) if Mp_n is the connected twofold covering group of Sp_n, with $\tilde{\sigma} \to \sigma$ as the covering homomorphism, then there exist unique unitary operators $r(\tilde{\sigma})$ such that (1) $r(\tilde{\sigma})\, vr(\tilde{\sigma})^{-1} = v \circ \sigma$ and (2) $\tilde{\sigma} \to r(\tilde{\sigma})$ is a representation. This is called the metaplectic representation. In this paper we give a computable explicit

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description of $r(\tilde{\sigma})$, working with the simply connected covering group, rather than the twofold cover. One should note that for a set of special σ, explicit formulas (up to a phase factor) are already given in [W]. These special elements include an open subset of Sp_n and generate Sp_n as a group, and this has turned out to be adequate for most applications (see also [Fo]). However, because of the fundamental nature of the metaplectic representation (it is known to provide a unifying framework for many applications in analysis, geometry, and number theory see the papers of Howe, [G-S], etc.), it seems desirable to get as explicit a description as possible. In fact various such accounts exist (see [Sou-1, L-V, P, Ho, Fo] and [Ba, Raw-Ro] for accounts based on the Fock model). The version presented here is closer in spirit to that of [W, L]. In this paper we restrict ourselves to the real case.

The starting point is that associated to a fixed symplectic basis, one can make a canonical choice of the operators $\sigma \to r(\sigma)$. This is recalled in Section 2 (see Proposition 2.3). The various properties of this choice make it possible to compute the multiplier $c(\sigma_1, \sigma_2)$ explicitly in terms of index of inertia of a triplet $V, \sigma_1^{-1}V, \sigma_2 \cdot V$ of Lagrangian spaces. These two were done in [RR] and since this was not published we have included a brief sketch of the proof for Proposition 2.3. The Maslov index of curves in the Lagrangian Grassmannian [A, Duis, G-S] enters the picture in the metaplectic representation only because of its connection with this index $\mathrm{Inert}(V, \sigma_1^{-1} \cdot V, \sigma_2 \cdot V)$ (see Section 2 for definitions). In Section 3, we present in some detail the construction and various properties of the simply connected group. Here the definition of the half-integer valued function $m(\tilde{\sigma})$ is made possible by the description of the double coset decomposition; the double cosets are parametrized by two integers j and k (see Proposition 3.7) and $m = (1/2)j + k$. In Theorem 4.1, the 2-cocycle $c(\sigma_1, \sigma_2)$ is expressed as a coboundary explicitly in terms of $m(\tilde{\sigma})$, giving the metaplectic representation. In Section 5, we consider the Maslov bundle. We show that this line bundle M is a homogeneous line bundle of the simply connected covering group, although as a C^∞ line bundle it is equivalent to the trivial line bundle. Its connection with the natural line bundle E, associated to the metaplectic representation and the half-form bundles is also clarified.

2. Notation and Preliminaries

2.1. Let (X, ω) be a symplectic vector space over $\mathbb{R}$ of dim $2n$ and $e_1, ..., e_n, e_1^*, ..., e_n^*$, a fixed symplectic basis, i.e., $\omega(e_i, e_j) = \omega(e_i^*, e_j^*) = 0$ and $\omega(e_i, e_j^*) = \delta_{ij}$. Let $\Lambda(X)$ denote the Lagrangian Grassmaniann and SpX, the symplectic group. For any $L \in \Lambda(X)$, P_L, N_L are subgroups of

SpX, defined as $P_L = \{g \in SpX: g \cdot L = L\}$, and $N_L = \{\sigma \in SpX: \sigma = id \text{ on } L\}$. We write $V = \Sigma Re_i$, $V^* = \Sigma Re_j^*$, and write $P = P_V$. For any $\sigma \in SpX$, let $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ be the matrix presentation of σ in the fixed symplectic basis. Then $\alpha, \beta, \gamma, \delta \in M(n, \mathbb{R})$ and satisfy (i) ${}^t\alpha\delta - {}^t\gamma\beta = I_n$ and (ii) ${}^t\alpha\gamma$, ${}^t\beta\delta$ are symmetric. We also write Σ_n for the set of all $n \times n$, real symmetric matrices. Then $N_V = \{u_\beta: \beta \in \Sigma_n\}$ and $N_{V^*} = \{v_\gamma: \gamma \in \Sigma_n\}$ where

$$u_\beta = \begin{pmatrix} I & \beta \\ 0 & I \end{pmatrix}, \qquad v_\gamma = \begin{pmatrix} I & 0 \\ \gamma & I \end{pmatrix}. \tag{2.1}$$

Also $P_V = \{g = \left(\begin{smallmatrix} \alpha & \beta \\ 0 & \delta \end{smallmatrix}\right): {}^t\alpha\delta = I_n, \alpha{}^t\beta$ is symmetric$\}$. Write $\tau_j \in SpX$, $j = 1, 2, ..., n$, where

$$\tau_j \cdot e_j = e_j^*, \qquad \tau_j \cdot e_j^* = -e_j \tag{2.2}$$

and τ_j fixes all others in the basis, then $\tau_j^4 = id$, $j = 1, ..., n$; $\tau_1, ..., \tau_n$ all commute and generate a finite group $W = \{\tau_1^{l_1} \cdot \tau_n^{l_n}: l_j \in \mathbb{Z}\}$. Note

$$\tau = \tau_1 \cdot \tau_2 \cdots \tau_n = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \tag{2.3}$$

For any subset $S \subset \{1, 2, ..., n\}$, we write

$$V_S = \Sigma_{j \in S} \mathbb{R} e_j, \qquad V_S^* = \Sigma_{j \in S} \mathbb{R} e_j^*, \qquad X_S = V_S + V_S^*, \qquad \tau_S = \prod_{j \in S} \tau_j. \tag{2.4}$$

Note $\tau\sigma\tau^{-1} = \left(\begin{smallmatrix} \delta & -\gamma \\ -\beta & \alpha \end{smallmatrix}\right) = ({}^t\sigma)^{-1}$. Let K be the centraliser of τ in $Sp(X)$. Then K is a maximal compact subgroup and $K = \{\sigma = \left(\begin{smallmatrix} \alpha & -\beta \\ \beta & \alpha \end{smallmatrix}\right): \alpha + i\beta$ is a unitary matrix$\}$. Note $K = Sp(X) \cap O(2n)$, where $O(2n)$ is the orthogonal group relative to the fixed basis. Clearly $W \subset K$.

2.2. The group $G = Sp(X)$ acts transitively on $\Lambda(X) \simeq G/P$. For each $L \in \Lambda(X)$, the subset

$$\Lambda_L(X) = \{L' \in \Lambda(X): L \cap L' = (0)\} \tag{2.5}$$

is open and the group N_L acts simply transitively on it, and the action converts it into a coordinate open set. Next group action on $\Lambda(X) \times \Lambda(X)$ also has only a finite number of orbits and all pairs L_1, L_2 such that $\dim L_1 \cap L_2 = k$ constitute a single orbit. Equivalently the subgroup P has only a finite number of orbits in $\Lambda(X)$ and these are $\Lambda_{V,k}(X) = \{L \in \Lambda(X): \dim L \cap V = n - k\}$. In particular if $\Omega_j = \{\sigma \in Sp(X): \sigma \cdot V \in \Lambda_{V,k}\} = \{\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in Sp(X): \text{rank } \gamma = j\}$, then Ω_j is a single P double coset and $\tau_S \in \Omega_j$ if $|S| = j$, since $\tau_S \cdot V = V_S^* + V_{S'}$, S' being the complement of S. Thus

$$Sp(X) = \bigcup_{j=0}^{n} \Omega_j = P.W.P. \tag{2.6}$$

2.3. Next the group action on $\Lambda(X) \times \Lambda(X) \times \Lambda(X)$ also gives only a finite number of orbits. The invariants defining these orbits will now be described. First consider L_j, $j = 1, 2, 3$, mutually transversal. Then $\mathrm{Inert}(L_1, L_2, L_3)$ (called index of inertia in [L]) is the signature of a quadratic space constructed as follows: let $x_j \in L_j$ be such that $x_1 + x_2 + x_3 = 0$. Then any one of the x_j's determines all the others uniquely and $\omega(x_1, x_2) = \omega(x_2, x_3) = \omega(x_3, x_1) = Q_j(x_j)$ $(j = 1, 2, 3)$. All the quadratis spaces (L_j, Q_j) are isometric and their signature is denoted by $\mathrm{Inert}(L_1, L_2, L_3)$. Two such triplets are in the same orbit if and only if their indices of inertia are the same. (See [L] and [G-S] or [LV]). For a general triplet let $F = L_1 \cap L_2 + L_2 \cap L_3 + L_3 \cap L_1$. Then F is isotropic and the images of L_j, say $\bar{L}_j$ in the symplectic vector space $F^{\perp}/F$, are mutually transversal. Define $\mathrm{Inert}(L_1, L_2, L_3)$ to be equal to $\mathrm{Inert}(\bar{L}_1, \bar{L}_2, \bar{L}_3)$. The two triplets L_j, L'_j $(j = 1, 2, 3)$ are in the same orbit iff

(i) $\dim L_1 \cap L_2 \cap L_3 = \dim(L'_1 \cap L'_2 \cap L'_3)$,

(ii) $\dim L_i \cap L_j = \dim L'_i \cap L'_j$ for all i, j, and

(iii) $\mathrm{Inert}(L_1, L_2, L_3) = \mathrm{Inert}(L'_1, L'_2, L'_3)$.

(For this and the lemma below see [RR].) This result can be easily proved by induction on $\dim X$ or by using the following

LEMMA 2.1. *Let $L_j \in \Lambda(X)$ be arbitrary. Then there exists orthogonal decomposition $X = \sum_{j=0}^{4} X_j$ into symplectic subspaces such that $L_j = \Sigma L_j \cap X_k$ and*

(1) *On X_0, $L_1 = L_2 = L_3$.*

(2) *On X_1, $L_2 = L_3$ and they both are transversal to L_1.*

(3) *On X_2, $L_3 = L_1$ and $L_1 \cap L_2 = (0)$.*

(4) *On X_3, $L_1 = L_2$ and $L_1 \cap L_3 = (0)$.*

(5) *On X_4, L_1, L_2, L_3 are mutually transversal.*

From these one deduces the following easily.

LEMMA 2.2. *Let $\sigma_1, \sigma_2 \in Sp(X)$ be arbitrary. Then there exist $g_1, g_2, g \in P$ and $\kappa_1, \kappa_2 \in SpX$ such that $g_1 \kappa_1 g = \sigma_1$, $g^{-1} \kappa_2 g_2 = \sigma_2$, where κ_1, κ_2 have the following special form: there exists a partition S_j $(0 \leqslant j \leqslant 4)$ of $\{1, 2, ..., n\}$ such that in the decomposition $X = \Sigma X_{S_j}$, one has*

$$\kappa_1 = \mathrm{diag}(I, \tau, I, \tau, v_\gamma), \qquad \kappa_2 = \mathrm{diag}(I, \tau, \tau, I, \tau).$$

Here the isometry class of γ is uniquely determined by the property that $\mathrm{sgn}\, \gamma = \mathrm{Inert}(V, \sigma_1^{-1} V, \sigma_2 V)$.

A proof is easily given on the basis of the observation that κ_1 and κ_2 are constructed so that the triplet V, $\sigma_1^{-1}V$, $\sigma_2 V$ and the triplet V, $\kappa_1^{-1}V$, $\kappa_2 V$ are in the same G-orbit (for detals see [RR]).

2.4. We next establish our notation about the metaplectic (projective) representation and also, a few facts about the construction of what we call the standard model of it (associated to a symplectic basis of X). References for this part are [Fo, L-V, G-S, Sou, L, H] and the original papers of [W, S]. For the Hiesenberg group $X \times \mathbb{R}$ with group law $(z, t)(z', t') = (z + z', t + t' + \frac{1}{2}\omega(z, z'))$, we use as the Schrodinger representation, the following: if $z = \Sigma p_j e_j + \Sigma q_j e_j^*$ or $z = (p, q)$, then

$$\nu(p, q, t) f \cdot (x) = e^{2\pi i t + 2\pi i p \cdot (x - (1/2)q)} f(x - q) \tag{2.7}$$

for all $f \in \mathscr{S}(R^n) = \mathscr{S}(V^*)$ the Schwartz space. (This is the Fourier transform of the representation ρ considered in [Fo].) Note we write $\nu(p, q) = \nu(p, q, 0)$. Now the Lie algebra of the Hersenberg group is $X \oplus \mathbb{R}$ with the identity map serving as the exponential map and the Lie bracket being $[(z, t), (z', t')] = (0, \omega(z, z'))$. Then the infinitesimal representation of ν is $\dot{\nu}$ and on the Schwartz space we have

$$\dot{\nu}(e_j) = 2\pi i x_j, \qquad \dot{\nu}(e_j^*) = -\frac{\partial}{\partial x_j}. \tag{2.8}$$

Now the group G acts as automorphisms of the Hersenberg group via $\sigma \cdot (z, t) = (\sigma \cdot z, t)$ and by the Stone–von Neumann theorem, the two representations ν and $\nu \circ \sigma$ are unitarily equivalent. These intertwining unitary operators are determined only up to a scalar multiple and one gets in this way a projective representation of G known as the metaplectic or oscillator projective representation. These intertwining operators where written down explicitly for special elements of G, for example, in [W] itself (or any of the other references mentioned earlier). We recall these. If $\sigma = \left(\begin{smallmatrix}\alpha & \beta \\ 0 & \delta\end{smallmatrix}\right) \in P$, then one can verify that the operator

$$r(\sigma) f \cdot (x) = |\det \delta|^{1/2} \exp(i\pi\beta'\alpha[x]) f(\delta^{-1} \cdot x) \tag{2.9}$$

intertwines ν and $\nu \circ \sigma$ or $r(\sigma)\, \nu(z)\, r(\sigma)^{-1} = \nu(\sigma \cdot z)$ and moreover $r(\sigma_1 \sigma_2) = r(\sigma_1)\, r(\sigma_2)$ for all $\sigma_1, \sigma_2 \in P$. Next similarly by direct verification, the intertwining operator corresponding to τ_j is $\mathscr{F}_j$—the partial Fourier transform corresponding to the jth variable, i.e., for any subset $S \subset \{1, 2, ..., n\}$, the partial Fourier transform $\mathscr{F}_S$ corresponding to the variables x_j, $j \in S$ is defined as

$$\mathscr{F}_S(f_1 \otimes f_2)(x_S + x_{S'}) = \mathscr{F} f_1 \cdot (x_S) \otimes f_2(x_{S'}). \tag{2.10}$$

where $f_1 \in \mathscr{S}(V_S^*)$, $f_2 \in \mathscr{S}(V_{S'}^*)$. The Fourier transform in $\mathscr{S}(R^n)$ is defined according to the formula

$$\mathscr{F}f \cdot (x) = \int_{R^n} e^{-2\pi i x \cdot y} f(y)\, dy. \tag{2.11}$$

If you now define for $\sigma = \tau_1^{k_1} \cdots \tau_n^{k_n} \in W$

$$r(\sigma) = \mathscr{F}_1^{k_1} \cdots \mathscr{F}_n^{k_n} \tag{2.12}$$

then $r(\sigma)\, v(z)\, r(\sigma)^{-1} = v(\sigma \cdot z)$ and $r(\sigma_1 \sigma_2) = r(\sigma_1)\, r(\sigma_2)$, if $\sigma_1, \sigma_2 \in W$. In particular $r(\tau) = \mathscr{F}$ is the Fourier transform on R^n (note $\tau = \tau_1 \cdot \tau_2 \cdots \tau_n$). Since $G = PWP$, the operators corresponding to other elements of G can be written down. This is how, for example, the operator for $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ with $\det \gamma \neq 0$ is written down in [W]. The following fact was observed n [RR].

PROPOSITION 2.3. *If $w, w' \in W$ and $g_1, g_2, g_1', g_2' \in P$ are such that $g_1 w g_2 = g_1' w' g_2'$, then*

$$r(g_1)\, r(w_1)\, r(g_2) = r(g_1')\, r(w')\, r(g_2'). \tag{2.13}$$

In particular this shows that there is a well defined and unique choice of intertwining operators $\sigma \to r(\sigma)$ such that $r(\sigma)$ is given by (2.9) when $\sigma \in P$, and by (2.10) when $\sigma \in W$ and has the property $r(g_1 \sigma g_2) = r(g_1)\, r(\sigma)\, r(g_2)$ when $g_1, g_2 \in P$ and $\sigma \in G$.

Since this fact is not mentioned in any of the above books and the article [RR] is not published we will give a brief sketch of its proof. It is easy to see (since any $w \in W$ is of the form $w = \tau_S \cdot g$ for some $g \in W \cap P$) that the verification of (2.13) reduces to checking the following two statements

(1) If S_1, S_2 are two subsets of $\{1, 2, ..., n\}$ with $|S_1| = |S_2|$ and ξ is a permutation of $\{1, 2, ..., n\}$ taking S_1 to S_2 and $g = \left(\begin{smallmatrix} \xi & 0 \\ 0 & \xi \end{smallmatrix}\right)$, then $g \in P$ and $g \tau_{S_1} g^{-1} = \tau_{S_2}$ and $r(g) \cdot r(\tau_{S_1}) = r(\tau_{S_2})\, r(g)$.

(2) If $g, g' \in P$ are such that $g\tau_S = \tau_S g'$, then

$$r(g)\, r(\tau_S) = r(\tau_S)\, r(g'). \tag{2.14}$$

Actual verification of (1) is straightforward. For (2) we need the following

LEMMA 2.4. *Let $P' = P \cap (\tau_S^{-1} P \tau_S)$. Then $g \in P'$ or both g and $g' = \tau_S^{-1} g \tau_S$ belong to P if and only if $g = \left(\begin{smallmatrix} \alpha & \beta \\ 0 & \delta \end{smallmatrix}\right)$ $(\in P)$ has the form (in the decomposition $V = V_S + V_{S'}$, $V^* = V_S^* + V_{S'}^*$, S' being the complement of S)*

$$\alpha = \begin{pmatrix} \alpha_{11} & 0 \\ \alpha_{21} & \alpha_{22} \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}, \qquad \delta = \begin{pmatrix} \delta_{11} & \delta_{12} \\ 0 & \delta_{22} \end{pmatrix}.$$

In this case $g' = \left(\begin{smallmatrix} \alpha' & \beta' \\ 0 & \delta' \end{smallmatrix}\right)$, *with* $\alpha' = \left(\begin{smallmatrix} \delta_{11} & 0 \\ \beta_{21} & \alpha_{22} \end{smallmatrix}\right)$

$$\beta' = \begin{pmatrix} 0 & \delta_{12} \\ -\alpha_{21} & \beta_{22} \end{pmatrix}, \qquad \delta' = \begin{pmatrix} \alpha_{11} & -\beta_{12} \\ 0 & \delta_{22} \end{pmatrix}.$$

In particular if $g = g_0 \cdot u_0$, *with* $g_0 = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \delta \end{smallmatrix}\right)$, $u_0 = \left(\begin{smallmatrix} I & \alpha^{-1}\beta \\ 0 & I \end{smallmatrix}\right)$ *then* $g \in P'$ *if and only if both* g_0 *and* $u_0 \in P'$. *Moreover*

$$\det(g \mid V) \det(g' \mid V) = \{\det(g \mid V_{S'})\}^2.$$

Proof of the Lemma. Observe that the matrix of τ_S in the decomposition $X = V_S + V_{S'} + V_S^* + V_{S'}^*$ is

$$\tau_S = \begin{pmatrix} 0 & 0 & -I & 0 \\ 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}. \tag{2.15}$$

If you write out the condition $\tau_{S'}^{-1} g \tau_{S'} \in P$, you get the formulas for g and g'. The others follows from these.

Coming back now to the proof of (2.14) we observe that this can be broken up into two parts according to when $g \in P'$ is of the form $g = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \delta \end{smallmatrix}\right)$ and g is of the form $= \left(\begin{smallmatrix} I & \beta \\ 0 & I \end{smallmatrix}\right)$. In each of these cases the equality (2.14) is easily checked by using a simple change of variables in integrals. We omit the details.

Remark 2.5. The choice $\sigma \to r(\sigma)$ of unitary operators introduced above, depends only on the symplectic basis and will be called the standard model associated to that basis. We note that this has the tensor product property, i.e., if $S_1, ..., S_l$ is a partition of $\{1, ..., n\}$ and the symplectic subspaces X_S, bases, etc. (see 2.4) are naturally defined and if r_S is the standard model associated to the data X_S, e_j, e_j^*, $j \in S$, then

$$r(\sigma)\,\varphi = r_{S_1}(\sigma_1)\,\varphi_1 \otimes r_{S_2}(\sigma_2)\,\varphi_2 \otimes \cdots \otimes r_{S_l}(\sigma_l)\,\varphi_l, \tag{2.16}$$

where $\sigma = \operatorname{diag}(\sigma_1, ..., \sigma_l)$, $\sigma_j \in Sp(X_{S_j})$, and $\varphi = \varphi_1 \otimes \cdots \otimes \varphi_l$, $\varphi_j \in \mathscr{S}(V_{S_j}^*)$. This follows easily from the fact that (2.16) is easily checked when $\sigma \in P$ or $\sigma \in W$.

2.5. *The multiplier* $c(\sigma_1, \sigma_2)$. The standard model $r(\sigma)$ is only a projective representation of $Sp(X)$ and so there exists scalars $c(\sigma_1, \sigma_2)$ such that $r(\sigma_1 \sigma_2) = c(\sigma_1, \sigma_2)\, r(\sigma_1)\, r(\sigma_2)$ for all $\sigma_1, \sigma_2 \in Sp(X)$. The following properties of the multiplier are obvious from the definition of the standard model.

$$c(g_1 \sigma_1 g, g^{-1} \sigma_2 g_2) = c(\sigma_1, \sigma_2) \qquad \text{for all} \quad \sigma_1, \sigma_2 \in Sp(X) \tag{2.17}$$

and $g_1, g_2, g \in P$.

$$c(\sigma_1, \sigma_2) = 1 \qquad \text{for all} \quad \sigma_1, \sigma_2 \in W. \tag{2.18}$$

If $S_1, S_2, ..., S_l$ is a partition of $\{1, 2, ..., n\}$ and $\sigma = \operatorname{diag}(\sigma_1, ..., \sigma_l)$, $\sigma' = \operatorname{diag}(\sigma'_1, ..., \sigma'_l)$ with $\sigma_j, \sigma'_j \in Sp(X_{S_j})$, then (2.19)

$$c(\sigma_1, \sigma_2) = \prod c_{S_j}(\sigma_j, \sigma'_j). \tag{2.20}$$

We recall here the main calculation, already done in [W].

$$c(v_\gamma, \tau) = \exp - \frac{i\pi}{4} \operatorname{sgn} \gamma, \tag{2.21}$$

where v_γ is defined in (2.1), τ in (2.3).

PROPROSITION 2.6. *The five properties,* (2.17)–(2.21), *determine the multiplier completely. In fact we have*

$$c(\sigma_1, \sigma_2) = \exp - \frac{i\pi}{4} \operatorname{Inert}(V, \sigma_1^{-1} V, \sigma_2 \cdot V),$$

where $\operatorname{Inert}(L_1, L_2, L_3)$ *is the index of inertia of a triplet of Lagrangian subspaces* (*see paragaph* 2.3).

This follows from Lemma 2.2 and the properties (2.17)–(2.21) of the multiplier (for details see [RR]). For another version of this result see [L-V].

3. The Simply Connected Covering Group

3.1. In this section we discuss in some detail the construction and properties of the simply connected covering group of $G = Sp(X)$. (For other presentations see [Ba, G-S, L-V] and possibly others.) The details developed here will be used in Sections 4–5. Let $\mathscr{H}_n$ denote the generalised upper half-plane $= \{Z = A + iB$: with A, B, real, $n \times n$ symmetric and B positive definite$\}$. Then $G = SpX$ acts transitively on $\mathscr{H}_n$, by the formula

$\sigma \cdot Z = (\alpha Z + \beta)(\gamma Z + \delta)^{-1}$, where $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$, and the stabilizer at iI_n is K. Let $j(\sigma, Z) = \det(\gamma Z + \delta)$. Then

$$j(\sigma_1 \sigma_2, Z) = j(\sigma_1, \sigma_2 \cdot Z)\, j(\sigma_2, Z). \tag{3.1}$$

Let $\Gamma_\sigma = \{\psi : \mathscr{H}_n \to \mathbb{C};\ \psi$ holomorphic and satisfies $\exp \psi(Z) = j(\sigma, Z)$ for all $Z\}$. Then note $\Gamma_{I_{2n}} = \{2\pi i l : l \in \mathbb{Z}\}$. Also in general if $\psi, \psi' \in \Gamma_\sigma$, then $\psi - \psi'$ is a constant $= 2\pi i l$, for some integer $l \in \mathbb{Z}$. Note also that the cocycle property (3.1) implies that if $\psi_j \in \Gamma_{\sigma_j}$, $j = 1, 2$, then $\psi = \psi_1 \circ \sigma_2 + \psi_2 \in \Gamma_{\sigma_1 \sigma_2}$. We now clearly have

Proposition 3.1. *Let $\tilde{G} = \{(\sigma, \psi) : \sigma \in G, \psi \in \Gamma_\sigma\}$. Then the map $(\sigma_1, \psi_1), (\sigma_2, \psi_2) \to (\sigma_1 \cdot \sigma_2, \psi_1 \circ \sigma_2 + \psi_2)$ is a group law on $\tilde{G}$, with $(I_{2n}, 0)$ as the identity element and $(\sigma^{-1}, -\psi \circ \sigma^{-1})$ as the inverse of (σ, ψ). Moreover $(\sigma, \psi) \to \sigma$ is a homomorphism onto G with kernel $\Gamma = \{(I_{2n}, 2\pi i l) : l \in \mathbb{Z}\}$. Thus as groups $\tilde{G}/\Gamma = G$.*

Next we topologize $\tilde{G}$ by giving it the subspace topology of the product $G \times \mathrm{Hol}(\mathscr{H}_n)$. Here $\mathrm{Hol}(\mathscr{H}_n)$ has the topology of uniform convergence on compact sets. Then $\tilde{G}$ is a topological group, Γ is a closed, discrete central subgroup, and $G = \tilde{G}/\Gamma$ as topological groups. From general theory, $\tilde{G}$ is also a Lie group with the same Lie algebra as G. We will next show that $\tilde{G}$ is connected. This needs some preparation. It is known that $\sigma = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \in K \to u(\sigma) = \alpha + i\beta \in U(n)$ is a group isomorphism. If $A = \begin{pmatrix} A_{11} & -A_{12} \\ A_{12} & A_{11} \end{pmatrix} \in$ Lie Algebra K, then ${}^tA_{11} = -A_{11}$ and ${}^tA_{12} = A_{12}$. Then $j(\exp A, iI_n) = \det u(\exp A) = \exp \operatorname{tr}(A_{11} + iA_{12}) = \exp i \operatorname{tr}(A_{12})$. Thus if $\psi \in \Gamma_{\exp A}$, then $\psi(iI_n) = i \operatorname{tr}(A_{12}) + 2\pi i l$ for some integer l. Thus there exists a unique element in $\Gamma_{\exp A}$, to be denoted by ψ_A such that $\psi_A(iI_n) = i \operatorname{tr} A_{12}$.

Lemma 3.2. *Let $A \in$ Lie alg K. Then the map $t (\in R) \to (\exp tA, \psi_{tA})$ is a continuous homomorphism. In particular $\exp_{\tilde{G}} A = (\exp A, \psi_A)$.*

Proof. It is sufficient to check that $t \to \tilde{\sigma}_t = (\exp tA, \psi_{tA})$ is a homomorphism. The rest follows from the general properties in Lie Groups. Now

$$\tilde{\sigma}_{t_1} \cdot \tilde{\sigma}_{t_2} = (\exp t_1 A \exp t_2 A, \psi_{t_1 A} \circ \exp t_2 A + \psi_{t_2 A}) = (\exp(t_1 + t_2) A, \psi)$$

say. Then $\psi \in \Gamma_{\exp(t_1 + t_2)A}$ and $\psi(iI_n) = \psi_{t_1 A}(iI_n) + \psi_{t_2 A}(iI_n) = i \operatorname{tr}((t_1 + t_2) A)$. Thus from the definition, $\psi = \psi_{(t_1 + t_2)A}$ and $\tilde{\sigma}_{t_1} \cdot \tilde{\sigma}_{t_2} = \tilde{\sigma}_{t_1 + t_2}$.

Proposition 3.3. *The group $\tilde{G}$ is connected and simply connected.*

Proof. Note G is connected. So to show that $\tilde{G}$ is connected it is sufficient to check that $\Gamma \subset$ connected component of the identity in $\tilde{G}$.

In fact we show that $\Gamma \subset \exp_{\tilde{G}}(\text{Lie alg } K)$. Let $H_D = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}$, where D is a diagonal matrix. Then $\exp 2\pi H_D = \begin{pmatrix} \cos 2\pi D & \sin 2\pi D \\ \sin 2\pi D & \cos 2\pi D \end{pmatrix} = I_{2n}$, if D has integral entries. Thus if D is an integral diagonal matrix, then $\exp_{\tilde{G}} 2\pi H_D = (I_{2n}, 2\pi i l)$, where $l = \operatorname{tr} D$. Thus $\Gamma \subset \exp_{\tilde{G}}(\text{Lie alg } K)$ and $\tilde{G}$ is connected. Thus $\tilde{G}$ is a connected covering group of G and Γ is isomorphic to a quotient of $\pi_1(G)$. Since $\pi_1(G) \simeq \mathbb{Z}$, it follows that $\tilde{G}$ is the simply connected covering group of G.

The same argument also gives the following.

LEMMA 3.4. (1) *Let $\tilde{K}$ be the inverse image of K in $\tilde{G}$. Then $\tilde{K}$ is connected and simply connected. The map $(\sigma, \psi)(\in \tilde{K}) \to \psi(iI_n)$ is a continuous homomorphism of $\tilde{K}$ onto $i\mathbb{R}$.*

(2) *If $\sigma_1, \sigma_2 \in K$ commute and $\tilde{\sigma}_1, \tilde{\sigma}_2$ are two elements sitting above σ_1, σ_2 in $\tilde{G}$, then $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ also commute.*

For the proof we just note that K fixes iI_n and this gives the homomorphism property. As for part (2), consider $(\sigma_1, \psi_1)(\sigma_2, \psi_2)$ $(\sigma_1, \psi_1)^{-1} = (\sigma_1 \sigma_2 \sigma_1^{-1}, \psi)$ say. Then $\psi = \psi_1 \circ \sigma_2 \circ \sigma_1^{-1} + \psi_2 \circ \sigma_1^{-1} - \psi_1 \circ \sigma_1^{-1}$. Evaluating at iI_n gives $\psi(iI_n) = \psi_2(iI_n)$ or $(\sigma_1 \sigma_2 \sigma_1^{-1}, \psi) = (\sigma_2, \psi_2)$.

3.2. We now consider the subgroup $\tilde{W}$—the inverse image of W in $\tilde{G}$. First consider the element $\tau = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \in W$. Then $j(\tau, Z) = \det Z$. Let $Z \to \operatorname{tr} \log Z$ be the unique holomorphic function on $\mathscr{H}_n$, such that $\exp(\operatorname{tr} \log Z) = \det Z$ and $\operatorname{tr} \log(iI_n) = in\pi/2$. (This function and its properties are discussed in the Appendix.) Then we define

$$\tilde{\tau} = (\tau, \operatorname{tr} \log) \in \tilde{G}. \tag{3.2}$$

LEMMA 3.5. *For any subset $S \subset \{1, 2, ..., n\}$, $j(\tau_S, Z) = \det Z_S$ where Z_S is the $S \times S$ submatrix of Z. In particular* $\operatorname{tr} \log Z_S \in \Gamma_{\tau_S}$ *and we define $\tilde{\tau}_S = (\tau_S, \operatorname{tr} \log Z_S) \in \tilde{G}$. In particular $\tilde{\tau}_j = (\tau_j, \log z_{jj})$, z_{jj} being the jth diagonal entry of Z.*

Proof. We regard Z as a linear transformation of $(V^*)^{\mathbb{C}}$ to $V^{\mathbb{C}}$ (the complexifications). Write $Z = \begin{pmatrix} Z_S & * \\ * & * \end{pmatrix}$ in the decompositions $V = V_S + V_{S'}$ and $V^* = V_S^* + V_{S'}^*$. Then from the matrix form (2.15) of τ_S, we get $j(\tau_S, Z) = \det Z_S$. The rest is clear.

COROLLARY 3.6. *For each j, $(\tilde{\tau}_j)^4 = (I_{2n}, 2\pi i)$—a generator of Γ. Moreover $(\tilde{\tau}_j)^2 = (\tau_j^2, i\pi) \in \tilde{P}$. Also the $\tilde{\tau}_j$'s commute and $\tilde{\tau}_S = \prod_{j \in S} \tilde{\tau}_j$. In particular*

$$\tilde{W} = \{(\tilde{\tau}_1)^{k_1} \cdots (\tilde{\tau}_n)^{k_n} : k_1, ..., k_n \in \mathbb{Z}\}. \tag{3.3}$$

Proof. Note $(\sigma, \psi) \to \psi(iI_n)$ is a homomorphism on $\tilde{K}$. Since $\tau_j^4 = I_{2n}$, and $\tau_j^2 \in P$, the statements about $(\tilde{\tau}_j)^4$ and $(\tilde{\tau}_j)^2$ follow. Clearly the $\tilde{\tau}_j$'s commute by Lemma 3.4 and by evaluating at iI_n, one establishes that $\tilde{\tau}_S$ is the product of the $\tilde{\tau}_j$'s with $j \in S$. Note $\tilde{W}$ is a commutative subgroup containing Γ, with image W in G and so is the inverse image of W.

Next consider $\tilde{P}$—the inverse image of P. Note the connected component P° of P is $= \{g = (\begin{smallmatrix} \alpha & \beta \\ 0 & \delta \end{smallmatrix}) \in G: \det \delta > 0\}$. Note for $g \in P, j(g, Z) = \det \delta$ is a constant. Thus if $(g, \psi) \in \tilde{P}$, then ψ is a constant and in fact $\psi = \log |\det \delta| + i\pi m$ where m is an integer such that $(-1)^m |\det \delta| = \det \delta$. Note that $(g, \psi) \to \psi \in \mathbb{R} + i\pi\mathbb{Z}$ is a homomorphism. In particular $m(\tilde{g}) = m = (1/\pi) \operatorname{Im} \psi$, $(\tilde{g} = (g, \psi))$ is also a homomorphism of $\tilde{P}$ with kernel $(\tilde{P})^c$—the connected component of $\tilde{P}$. In fact $(\tilde{P})^c = \{\tilde{g} = (g, \log \det \delta): g \in P^c\}$. Clearly $\tilde{P}$ is the semidirect product of $(\tilde{P})^c$ and a discrete subgroup isomorphic to $\mathbb{Z}$. In fact $m(\tilde{\tau}_j^{2k}) = k$, so that if $\Gamma' = \{\tilde{\tau}_1^{2k}: k \in \mathbb{Z}\}$, then $\Gamma \subset \Gamma'$ and $\tilde{P} = (\tilde{P})^c \cdot \Gamma'$.

3.3. We now consider double coset decompositions. Let $\tilde{\Omega}_j$ be the inverse image of Ω_j (see 2.6). Then

$$\tilde{\Omega}_j = \tilde{P}\tau_S\tilde{P} \text{ if } |S| = j \qquad \text{and} \qquad \tilde{G} = \bigcup_{j=0}^{n} \tilde{\Omega}_j. \tag{3.4}$$

PROPOSITION 3.7. *Let $\tilde{\Omega}_{j,m} = (\tilde{P})^c\, \tilde{\tau}_S \tilde{\tau}_1^{2m} (\tilde{P})^c$. Then $\tilde{\Omega}_{j,m_1} \cap \tilde{\Omega}_{j,m_2} \neq \varnothing$ if and only if $m_1 = m_2$. Moreover $\tilde{\Omega}_j = \bigcup_{-\infty < m < \infty} \tilde{\Omega}_{j,m}$ is the decomposition of $\tilde{\Omega}_j$ into its connected components, each $\tilde{\Omega}_{j,m}$ being open in $\tilde{\Omega}_j$.*

Proof. For the first part it is sufficient to check the following. Suppose $\tilde{g}, \tilde{g}_1 \in (\tilde{P})^c$ and

$$\tilde{g}\tilde{\tau}_S\tilde{\tau}_1^{2m_1}\tilde{g}_1^{-1} = \tilde{\tau}_S\tilde{\tau}_1^{2m_2}, \qquad \text{then} \quad m_1 = m_2. \tag{3.5}$$

Clearly the equality (3.5) implies that

$$\tilde{g} \cdot \tilde{\tau}_S \cdot \tilde{g}'^{-1} = \tilde{\tau}_S, \qquad \text{where} \quad \tilde{g}' = \tilde{\tau}_1^{2m_2}\tilde{g}_1\tilde{\tau}_1^{-2m_1}. \tag{3.6}$$

Since $(\sigma, \psi) \to \psi$ is a homomorphism of $\tilde{P}$ into $\mathbb{C}$, we have

$$(\tilde{g}')^{-1} = ((g')^{-1}, im_1\pi - \log \det(g_1 \mid V) - im_2\pi) \tag{3.7}$$

(note $g, g_1 \in P^c$ and so $\det(g_1 \mid V) > 0$). Now the relation (3.6), using the multiplication law in the group, becomes $g\tau_S g'^{-1} = \tau_S$ and

$$\begin{aligned} &\log(\det(g \mid V)) + \operatorname{tr} \log(g'^{-1} \cdot Z)_S - \log(\det(g_1 \mid V)) + i\pi(m_1 - m_2) \\ &\quad = \operatorname{tr} \log Z_S. \end{aligned}$$

We thus have

$$(m_1 - m_2)\pi = \text{Imag. part of } \{\operatorname{tr}\log Z_S - \operatorname{tr}\log(g'^{-1}\cdot Z)_S\}.$$

We now compute $(g'^{-1}\cdot Z)_S$. Now $g' = \left(\begin{smallmatrix}\alpha' & \beta' \\ 0 & \delta'\end{smallmatrix}\right)$ implies

$$\begin{aligned} g'^{-1}\cdot Z &= ({}^t\delta'\cdot Z - {}^t\beta')({}^t\alpha')^{-1} = ({}^t\delta'\cdot Z - {}^t\beta')\cdot\delta' \\ &= {}^t\delta'(Z\delta' - \beta'), \end{aligned} \tag{3.8}$$

where in the last step we have used symmetry of Z and $g'^{-1}\cdot Z$. Now $g\tau_S g'^{-1} = \tau_S$ and so we can use Lemma 2.4 and if $Z = \left(\begin{smallmatrix}Z_S & * \\ * & *\end{smallmatrix}\right)$ in the decompositions $V = V_S + V_{S'}$, $V^* = V_S^* + V_{S'}^*$, we get

$$g'^{-1}\cdot Z = \begin{pmatrix} {}^t\alpha_{11} & 0 \\ * & * \end{pmatrix}\begin{pmatrix} Z_S & * \\ * & * \end{pmatrix}\begin{pmatrix} \alpha_{11} & * \\ 0 & * \end{pmatrix} + \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}.$$

Thus $(g'^{-1}\cdot Z)_S = {}^t\alpha_{11}\cdot Z_S\alpha_{11}$. From the property of tr log (see Appendix), $\operatorname{tr}\log({}^t\alpha_{11}Z_S\alpha_{11}) = \operatorname{tr}\log Z_S + \log(\det\alpha_{11})^2$. From this we get $m_1 = m_2$. Note for the second part $\tilde{\Omega}_j = \tilde{P}\tilde{\tau}_S\tilde{P} = (\tilde{P})^\circ\tau_S\Gamma'(\tilde{P})^\circ$, where $\Gamma' = \{\tilde{\tau}_1^{2m} : m\in\mathbb{Z}\}$. Thus $\tilde{\Omega}_j$ is the union of $\tilde{\Omega}_{j,m}$. From general theory at least one of the $(\tilde{P})^\circ$ double cosets in $\tilde{\Omega}_j$ is open. All these double cosets are homeomorphic since $\tilde{\Omega}_{j,m} = \tilde{\Omega}_{j,0}\tilde{\tau}_1^{2m}$. These facts imply that $\tilde{\Omega}_{j,m}$ are precisely the connected components. ∎

Remark 3.8. The double coset $(\tilde{P})^\circ\tilde{\tau}_S\tilde{\tau}_1^{2m}(\tilde{P})^\circ$ depends only on $|S| = j$, rather than on the particular set S. To see this, note that there exists a $\xi\in SO(n)$ such that $g\tau_{S_1}g^{-1} = \tau_{S_2}$ if $|S_1| = |S_2|$, where $g = \left(\begin{smallmatrix}\xi & 0 \\ 0 & \xi\end{smallmatrix}\right)$. Then $\tilde{g} = (g, 0)\in(\tilde{P})^\circ$ and $\tilde{g}\tilde{\tau}_{S_1}\tilde{g}^{-1} = \tilde{\tau}_{S_2}$.

DEFINITION 3.9. (1) For any $\tilde{\sigma} = (\sigma, \psi)\in\tilde{G}$, define $j(\tilde{\sigma}) = j$ if $\tilde{\sigma}\in\tilde{\Omega}_j$ or equivalently $j(\tilde{\sigma}) = n - \dim(\sigma\cdot V\cap V)$.

(2) Define $m(\tilde{\sigma}) = (1/2)j + k$ if $\tilde{\sigma}\in\tilde{\Omega}_{j,k}$.

Note $j(\tilde{\sigma}) = 0$ if and only if $\tilde{\sigma}\in\tilde{P}$. Also if $\tilde{g} = (g, \psi)\in\tilde{P}$, then $m(\tilde{g}) = (1/\pi)\operatorname{Im}\psi$ is integer valued homomorphism of $\tilde{P}$. In addition we have

$$j(\tilde{g}_1\tilde{\sigma}\tilde{g}_2) = j(\tilde{\sigma}), \qquad m(\tilde{g}_1\tilde{\sigma}\tilde{g}_2) = m(\tilde{g}_1) + m(\tilde{\sigma}) + m(\tilde{g}_2) \tag{3.9}$$

if $\tilde{g}_1, \tilde{g}_2\in\tilde{P}$. Moreover if $\sigma = \tilde{\tau}_1^{k_1}\cdots\tilde{\tau}_n^{k_n}$, then $j(\tilde{\sigma}) = |S|$, where $S = \{j : k_j \text{ is odd}\}$ and

$$m(\tilde{\sigma}) = \frac{1}{2}(k_1 + \cdots + k_n) = \frac{1}{\pi}\operatorname{Im}\psi(iI_n), \tag{3.10}$$

if $\tilde{\sigma} = (\sigma, \psi)$. The last step follows from the observation $\tilde{\sigma} = \tilde{\tau}_S\tilde{\tau}_1^{k'_1}\cdots\tilde{\tau}_n^{k'_n} = \tilde{\tau}_S\tilde{\tau}_1^{l}\tilde{g}$, with $\tilde{g}\in(\tilde{P})^\circ$, where $l = k'_1 + \cdots + k'_n$, each k'_i being even.

3.4. The above construction of a simply connected covering group G depended on the choice of a fixed symplectic basis. Now we discuss how this construction behaves under direct sums. For this purpose we identity $\mathscr{H}_n$ as linear maps of $(V^*)^{\mathbb{C}}$ to $V^{\mathbb{C}}$ relative to the bases $e_1^*, ..., e_n^*$ and $e_1, ..., e_n$. (See Section 2). This is motivated by the geometrical identification of the generalized upper half plane $\mathscr{H}_n$ with the set of (strictly) positive Lagrangian supspaces of $X^{\mathbb{C}}$. In fact this identification map is $Z \to u_Z \cdot V^*$, where $u_Z = \left(\begin{smallmatrix} I & Z \\ O & I \end{smallmatrix}\right) \in Sp(X^{\mathbb{C}})$. Note if $L_Z = u_Z \cdot V^*$, then $\sigma \cdot L_Z = L_{\sigma \cdot Z}$ for all $\sigma \in Sp(X)$. Now suppose $S_1, ..., S_k$ is a partition of $\{1, 2, ..., n\}$. Let $|S_j| = n_j$, $j = 1, 2, ..., k$. Thus $\mathscr{H}_{n_j} \subset \mathrm{Hom}(V_{S_j}^*, V_{S_j})$. Then there is a natural map

$$\mathscr{H}_{n_1} \times \cdots \times \mathscr{H}_{n_k} \to \mathscr{H}_n, \qquad (Z_1, ..., Z_k) \to Z$$

such that $Z \mid V_{S_j}^* = Z_j$. In this case we write $Z = \mathrm{diag}(Z_1, ..., Z_k)$. If $\sigma = \mathrm{diag}(\sigma_1, ..., \sigma_k)$ with $\sigma_j \in Sp(X_{S_j})$, then one checks easily that $\sigma \cdot \mathrm{diag}(Z_1, ..., Z_k) = \mathrm{diag}(\sigma_1 \cdot Z_1, ..., \sigma_k \cdot Z_k)$. Moreover,

$$j(\sigma, \mathrm{diag}(Z_1, ..., Z_k)) = \prod_{i=1}^{k} j(\sigma_i, Z_i).$$

LEMMA 3.10. *With notation as above, let $\tilde{\sigma}_j \in \tilde{G}_{S_j}$. Suppose $\tilde{\sigma}_j = (\sigma_j, \psi_j)$, $j = 1, 2, ..., k$. Let $\sigma = \mathrm{diag}(\sigma_1, ..., \sigma_k)$. Then there exists a unique $\psi \in \Gamma_\sigma$, such that $\psi(\mathrm{diag}(Z_1, ..., Z_k)) = \psi(Z_1) + \cdots + \psi(Z_k)$. We write this (σ, ψ) as $\mathrm{diag}(\tilde{\sigma}_1, ..., \tilde{\sigma}_n)$.*

Proof. Since $j(\sigma_j, Z_j) = \exp \psi_j(Z_j)$, it follows that $j(\sigma, iI_n) = j(\sigma_1, iI_{S_1}) \cdots j(\sigma_k, iS_k) = \exp \Sigma\psi_j(iI_{S_j})$. Thus there exists a unique $\psi \in \Gamma_\sigma$, such that $\psi(iI_n) = \Sigma\psi_j(iI_{S_j})$. From this one deduces easily that $\psi(\mathrm{diag}(Z_1, ..., Z_k)) = \psi(Z_1) + \cdots + \psi(Z_k)$. In fact exponentials of both sides in the above equation agree and they both agree at one point. ∎

For any subset S, we have the symplectic subspace X_S with symplectic basis $\{e_j, e_j^*, j \in S\}$ and so we have the groups $G_S = SpX_S$, $\tilde{G}_S$, P_S, $\tilde{P}_S$, W, $\tilde{W}_S$, etc. Also the functions $j_S(\cdot)$, $m_S(\cdot)$, are defined analogously on $\tilde{G}_S$.

PROPOSITION 3.11. *Let $\tilde{\sigma}_j \in \tilde{G}_{S_j}$, $S_1, ..., S_k$ being a partition of $\{1, 2, ..., n\}$. Let $\tilde{\sigma} = \mathrm{diag}(\tilde{\sigma}_1, ..., \tilde{\sigma}_k)$. Then if f is any of the functions $j(\cdot)$, $m(\cdot)$ on $\tilde{G}$ then $f(\tilde{\sigma}) = f_{S_1}(\tilde{\sigma}_1) + \cdots + f_{S_k}(\tilde{\sigma}_k)$.*

Proof. Note that if $\tilde{\sigma} = (\sigma, \psi)$, then $\sigma = \mathrm{diag}(\sigma_1, ..., \sigma_k)$ so that $\dim \sigma V \cap V = \Sigma \dim \sigma_j \cdot V_{S_j} \cap V_{S_j}$ or $j(\tilde{\sigma}) = \Sigma j_{S_i}(\tilde{\sigma}_i)$. Next if $\tilde{\sigma} \in \tilde{P}$, then $\tilde{\sigma} = (\sigma, \psi)$ with ψ constant and $= \Sigma\psi_j$ where ψ_j is also constant, since $\tilde{\sigma}_j = (\sigma_j, \psi_j) \in \tilde{P}_{S_j}$. From this it follows that $m(\tilde{\sigma}) = \Sigma m_{S_j}(\tilde{\sigma}_j)$ if $\tilde{\sigma} \in \tilde{P}$. Note $\tilde{\sigma} \in \tilde{P}$ implies $\tilde{\sigma}_j \in \tilde{P}_{S_j}$. Thus in view of (3.9), it is sufficient to check this additivity property when $\tilde{\sigma}_j \in \tilde{W}_{S_j}$ for all j. In this case using (3.10), we have $m(\tilde{\sigma}) = (1/2)\, \psi(iI_n) = (1/2)\, \Sigma\psi_j(iI_{S_j}) = \Sigma m_{S_j}(\tilde{\sigma}_j)$. ∎

LEMMA 3.12. *Let $\gamma \in \Sigma_n$ be a $n \times n$, symmetric real matrix with* $\det \gamma \neq 0$. *Let $v_\gamma = \begin{pmatrix} I & 0 \\ \gamma & I \end{pmatrix}$ and define $\tilde{v}_\gamma = \tilde{\tau}\tilde{u}_{-\gamma}\tilde{\tau}^{-1}$, where $u_\gamma = \begin{pmatrix} I & \gamma \\ 0 & I \end{pmatrix}$, $\tilde{u}_\gamma = (u_\gamma, 0)$, and $\tilde{\tau} = (\tau, \operatorname{tr} \log)$. Then $m(\tilde{v}_\gamma) = (1/2) \operatorname{sgn} \gamma = (n^+(\gamma) - n^-(\gamma))/2$ where $n^+(\gamma)$ is the number of positive (negative) eigenvalues of γ. The same formula holds for all γ.*

Proof. Suppose $\tilde{v}_\gamma = (v_\gamma, \psi_\gamma)$, then from Proposition 3.1 it follows that $\psi_\gamma(Z) = \operatorname{tr} \log(-Z^{-1} - \gamma) - \operatorname{tr} \log(-Z^{-1})$. Next $v_\gamma = g \cdot \tau \cdot u_{\gamma^{-1}}$, where $g = \begin{pmatrix} \gamma^{-1} & I \\ 0 & \gamma \end{pmatrix}$. Let $\tilde{g}$ denote some element in $\tilde{G}$ above g. Then $\tilde{v}_\gamma = \tilde{g} \cdot \tilde{\tau} \cdot \tilde{u}_{\gamma^{-1}}$ $(I_{2n}, 2\pi i l)$ for some integer l. Thus $m(\tilde{v}_\gamma) = m(\tilde{g}) + m(\tilde{\tau})\, 2l = (1/2)\, n + k + 2l$ where $\tilde{g} = (g, \log |\det \gamma| + i\pi k)$. Now

$$\tilde{g}\tilde{\tau}u_{\gamma^{-1}} = (v_\gamma, \log |\det \gamma| + i\pi k + \operatorname{tr} \log(Z + \gamma^{-1})).$$

Thus $\psi_\gamma(Z) - \operatorname{tr} \log(Z + \gamma^{-1}) = \log |\det \gamma| + i\pi(k + 2l)$. From the result in the Appendix (Proposition A.2) we get $k + 2l = -n^-(\gamma)$. ∎

Finally we note the following calculations of $m(\tilde{\sigma})$. The proofs are omitted.

LEMMA 3.13. (1) *Let $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with* $\det \gamma \neq 0$. *Let $\tilde{\sigma} = (\sigma, \psi) \in \tilde{G}$. Then $\psi(Z) = \log |\det \gamma| + \operatorname{tr} \log(Z + \gamma^{-1}\delta) + i\pi k$, for some integer k, such that $(-1)^k |\det \gamma| = \det \gamma$. Moreover, $m(\tilde{\sigma}) = (1/2)\, n + k$.*

(2) *Suppose instead* $\det \delta \neq 0$. *If $\tilde{\sigma} = (\sigma, \psi) \in \tilde{G}$, then $\psi(Z) = \log |\det \delta| + i\pi k + \operatorname{tr} \log(-Z^{-1} - \delta^{-1}\gamma) - \operatorname{tr} \log(-Z^{-1})$, where the integer k is such that $(-1)^k |\det \delta| = \det \delta$. Moreover $m(\tilde{\sigma}) = k + (1/2) \operatorname{sgn}(\delta^{-1}\gamma)$.*

4. THE MAIN THEOREM

The main result in this section is the lifting of the projective representation $\sigma \to r(\sigma)$ of G, by an explicit formula, to an ordinary representation of $\tilde{G}$.

THEOREM 4.1. *Let $b(\tilde{\sigma}) = \exp -(i\pi/2)\, m(\tilde{\sigma})$. Then*

$$c(\sigma_1, \sigma_2) = b(\tilde{\sigma}_1)\, b(\tilde{\sigma}_2)(b(\tilde{\sigma}_1\tilde{\sigma}_2))^{-1}.$$

In particular $\tilde{\sigma} \to r(\tilde{\sigma}) = b(\tilde{\sigma})\, r(\sigma)$ is a representation of $\tilde{G}$.

Proof. Let $h(\tilde{\sigma}_1, \tilde{\sigma}_2) = m(\tilde{\sigma}_1) + m(\tilde{\sigma}_2) - m(\tilde{\sigma}_1\tilde{\sigma}_2)$. Then the property (3.9) implies that $h(\tilde{\sigma}_1, \tilde{\sigma}_2)$ actually depends only on σ_1, σ_2, rather than on the elements $\tilde{\sigma}_1, \tilde{\sigma}_2$ sitting above them in $\tilde{G}$. So we write $h(\sigma_1, \sigma_2) = h(\tilde{\sigma}_1, \tilde{\sigma}_2)$. Next the same property also implies that $h(g_1\sigma_1 g, g^{-1}\sigma_2 g_2) = h(\sigma_1, \sigma_2)$ for all $g_1, g_2, g \in P$. Next the additivity property (Proposition 3.11) of $m(\cdot)$ implies the additivity property of h, i.e., if $S_1, ..., S_k$ is a partition of $\{1, 2, ..., n\}$, $\sigma_1 = \mathrm{diag}(\sigma_{11}, ..., \sigma_{1k})$ and $\sigma_2 = \mathrm{diag}(\sigma_{21}, ..., \sigma_{2k})$, then $h(\sigma_1, \sigma_2) = \Sigma h(\sigma_{1j}, \sigma_{2j})$. Next the function $m(\cdot)$, being a homomorphism on $\tilde{W}$, it follows that $h(\sigma_1, \sigma_2) = 0$ if $\sigma_1, \sigma_2 \in W$. Now suppose $h'(\sigma_1, \sigma_2) = (1/2)\,\mathrm{Inert}(V, \sigma_1^{-1} \cdot V, \sigma_2 \cdot V)$. Then $h'(\sigma_1, \sigma_2)$ has the same properties (see Section 2). Thus from Lemma 2.2, it would follow that $h(\sigma_1, \sigma_2) = h'(\sigma_1', \sigma_2')$ if we show that $h(v_\gamma, \tau) = h'(v_\gamma, \tau)$. Now from Lemma 3.12, $m(\tilde{v}_\gamma) = (1/2)\,\mathrm{sgn}\,\gamma$, $m(\tilde{v}_\gamma \tilde{\tau}) = m(\tilde{\tau} \cdot \tilde{u}_{-\gamma}) = m(\tilde{\tau})$, since $\tilde{u}_{-\gamma} \in (\tilde{P})^\circ$. Thus $h(v_\gamma, \tau) = (1/2)\,\mathrm{sgn}\,\gamma$. On the other hand, one checks directly from the definition of index of inertia that $h'(v_\gamma, \tau) = (1/2)\,\mathrm{sgn}\,\gamma$. Thus $h(\sigma_1, \sigma_2) = h'(\sigma_1, \sigma_2)$ for all σ_1, σ_2. The theorem now follows from Proposition 2.6.

Remark. (1) If $\sigma = (I_{2n}, 2\pi i l) \in \Gamma$, then $m(\tilde{\sigma}) = 2l$ and $\tau(\tilde{\sigma}) = (-1)^l\, id$. This gives the well known fact that the metaplectic representation is actually a representation of the 2-fold cover $\tilde{G}/\Gamma^\circ$, $\Gamma^\circ = \{I_{2n}, 2\pi i l) : l \text{ even}\}$, of G.

(2) If $\tilde{g} = (g, \psi) \in \tilde{P}$, $m(\tilde{g}) = (1/\pi)\,\mathrm{Im}\,\psi$, and

$$r(\tilde{g})\, f \cdot (x) = e^{-(1/2)\psi} (\exp i\pi\beta'\alpha \cdot [x])\, f(\delta^{-1} \cdot x), \qquad \text{if} \quad g = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix},$$

and similar formulas can be written down for $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, $\det \gamma \neq 0$, etc.

(3) Note the additivity property of m over direct sums (see Proposition 3.11) implies a tensor product property for the representation $\tilde{\sigma} \to r(\tilde{\sigma})$. (See Remark 2.5.)

(4) The proof of Theorem 4.1 actually gives that

$$\mathrm{Inert}(\sigma_1 \cdot V, \sigma_2 V, \sigma_3 \cdot V) = 2\{m(\tilde{\sigma}_2^{-1}\tilde{\sigma}_1) + m(\tilde{\sigma}_1\tilde{\sigma}_3) - m(\tilde{\sigma}_2^{-1}\tilde{\sigma}_3)\}.$$

Since $\mathrm{Inert}(\sigma \cdot L_1, \sigma \cdot L_2, \sigma \cdot L_3) = \mathrm{Inert}(L_1, L_2, L_3)$ for all $\sigma \in G$, it follows that

$$\begin{aligned} \mathrm{Inert}(\sigma_1 \cdot V, \sigma_2 \cdot V, \sigma_3 \cdot V) &= \mathrm{Inert}(V, \sigma_1^{-1}\sigma_2 \cdot V, \sigma_1^{-1}\sigma_3 \cdot V) \\ &= 2\{m(\tilde{\sigma}_2^{-1}\tilde{\sigma}_1) + m(\tilde{\sigma}_1^{-1}\tilde{\sigma}_3) - m(\tilde{\sigma}_2^{-1}\tilde{\sigma}_3)\}. \end{aligned}$$

In this connection compare Leray [L, Chap. 2].

We note next a simple formula for $r(\tilde{\sigma})\, F_Z$, where $F_Z(x) = \exp i\pi Z[x]$, $Z \in \mathscr{H}_n$. Up to an ambiguity in the phase factor, this is discussed in [Fo, p. 202]. (See the references cited there.) In our set up, this takes a simple form.

PROPOSITION 4.2. *For any $\tilde{\sigma} = (\sigma, \psi) \in \tilde{G}$ and $Z \in \mathscr{H}_n$*

$$r(\tilde{\sigma})\, F_Z = e^{-(1/2)\psi(Z)} F_{\sigma \cdot Z}.$$

Proof. The formula is easily checked for $\tilde{\sigma} \in \tilde{P}$ and $\tilde{\sigma} = \tilde{\tau}$. Also both sides are additive over direct sums, so the formula is valid for $\tilde{\sigma} = \tilde{\tau}_i$. Also the validity of (4.1) for $\tilde{\sigma}_1, \tilde{\sigma}_2$ implies its validity for $\tilde{\sigma}_1 \tilde{\sigma}_2$. So it is valid for all $\tilde{\sigma}$.

Remark. This proposition gives the well known fact that if $\tilde{\sigma} \in \tilde{K}$, $\tau(\tilde{\sigma})\varphi_0 = e^{-(1/2)\psi(iI_n)}\varphi_0$, where $\varphi_0 = F_Z$, with $Z = iI_n$. Also the matrix entries $a(\tilde{\sigma}) = (r(\tilde{\sigma}) F_{Z_1}, F_{Z_2})$ can be easily evaluated and in fact $= \{\exp(-(1/2)\psi(Z_1))\}\{\det(-i(\sigma \cdot Z_1 - \bar{Z}_2))\}^{-1/2}$. Various other evaluations are possible, but we do not pursue it here.

5. The Maslov Bundle

We begin with some remarks on notation, etc., on homogeneous line bundles. Let G be a Lie group, H a closed subgroup, and χ a quasi character of H, i.e., $\chi \in \mathrm{Hom}(H, \mathbb{C}^\times)$. Then the homogeneous line bundle over G/H, associated to χ may be constructed as follows: let $G \times_\chi \mathbb{C}$ denote the set of H orbits in $G \times \mathbb{C}$, with H acting on the right by (g, z), $h \to (gh, \chi(h)^{-1} z)$. Let $(g, z)H$ denote the H-orbit of (g, z). Let $\pi\colon (g, z)H = gH \in G/H$. With quotient C^∞-structure on $G \times_\chi \mathbb{C}$, it is a smooth line bundle over G/H. Then G acts on $G \times_\chi \mathbb{C}$, via $g_1, (g, z)H \to (g_1 g, z)H$ and this action is equivariant with the natural action of G on G/H. This construction gives a bijection between homogeneous line bundles on G/H, and quasi characters of H. If $\{U_i\}$ is a contractible open covering of G/H, with $\kappa_i\colon U_i \to G$ smooth cross-sections, then s_i: $xH \to (\kappa_i(xH), 1)H$ is a smooth local section over U_i, and the transition functions $c_{ij}(xH)$ are computed as $c_{ij}(xH) = \chi(\kappa_i^{-1}(xH)\kappa_j(xH))$. Note $s_j = c_{ij} \cdot s_i$ on $U_i \cap U_j$. As a homogeneous line bundle, $G \times_\chi \mathbb{C}$ is trivial if and only if χ is trivial, although as a (smooth) line bundle over G/H, it may be equivalent to a trivial bundle, even when χ is not trivial. In this connection we note the following.

Lemma 5.1. *If G/H is compact and there exists $\eta \in \mathrm{Hom}(H, \mathbb{C})$ such that $\chi = e^\eta$, then $G \times_\chi \mathbb{C}$ is equivalent to a trivial line bundle.*

Proof. Let $\varphi \in C_c^\infty(G)$, such that $\int_H \varphi(xh)\, d_l h = 1$ for all $x \in G$. (This is possible since G/H is compact.) Here $d_l h$ stands for left invariant Haar measure on H (see Helgason [H]). Let

$$f(x) = \int_H \varphi(xh)\, \eta(h)\, d_l h.$$

Then $f(xh) = f(x) - \eta(h)$ and if you write $a_i = \exp(-f \circ \kappa_i)$ on U_i, then one checks that $c_{ij} = a_i a_j^{-1}$. Since the a_i are smooth, this implies that the line bundle is trivial.

We next give the natural line bundle associated to the metaplectic representation. Using the notations introduced earlier, define for each $\lambda \in \Lambda(X)$, the space E_λ of tempered distributions, $E_\lambda = \{u \in S'(R^n)$: $\dot{v}(z)\,u = 0$ for all $z \in \lambda\}$. Here v is the Schrodinger representation and $\dot{v}$ is the corresponding representation of the Lie algebra of the Hersenberg group. Thus, for example, one checks easily that $E_V = \mathbb{C}\delta_0$; here δ_0 is the Dirac distribution at the origin. From $r(\sigma)\,\dot{v}(z)\,r(\sigma)^{-1} = \dot{v}(\sigma\dot{z})$, for $\sigma \in Sp(X)$, $z \in X$, it follows that $E_{\sigma\cdot\lambda} = r(\sigma)\,E_\lambda$ and so $E_{\sigma\cdot V} = \mathbb{C}r(\sigma)\,\delta_0$, $r(\sigma)$ being the projective representation of SpX, introduced in Section 2. Thus $\dim E_\lambda = 1$ for all $\lambda \in \Lambda(X)$. If $\tilde{\sigma} \to r(\tilde{\sigma})$ denotes the metaplectic representation (see Section 4), then we have seen that for $\tilde{g} = (g, \psi) \in \tilde{P}$ (note ψ is a constant), $r(\tilde{g}) = (\exp - (1/2)\operatorname{Im}\psi)\,r(g)$ and the formula for $r(g)$, $g \in P$ gives $r(g)\,\delta_0 = |\det \delta|^{-1/2}\,\delta_0$ if $g = \left(\begin{smallmatrix}\alpha & \beta\\ 0 & \delta\end{smallmatrix}\right)$. Since $\operatorname{Re}\psi = \log|\det\delta|$, it follows that

$$r(\tilde{g})\,\delta_0 = \chi(\tilde{g})\,\delta_0, \qquad \text{where} \quad \chi(\tilde{g}) = e^{-(1/2)\psi}. \tag{5.1}$$

Note that $(g, \psi) = \tilde{g} \to \psi$ is a homomorphism of $\tilde{P}$ into $\mathbb{C}$ and χ is a quasi character of $\tilde{P}$.

Let $E = \bigcup E_\lambda = \{(\lambda, u): \lambda \in \Lambda(X), u \in E_\lambda\}$. Now $\Lambda(X) = \tilde{G}/\tilde{P}$. Then the map $\Phi: \tilde{G} \times_\chi \mathbb{C} \to E$, defined as $\Phi: (\tilde{\sigma}, z)\,\tilde{P} \to (\sigma\cdot V, zr(\tilde{\sigma})\,\delta_0) \in E$, is a bundle map, linear on fibers. It is also a bijection. We will endow E with the structure of a smooth line bundle over $\Lambda(X)$ by requiring Φ be an isomorphism. Note Φ also intertwines $\tilde{G}$-action, with the natural $\tilde{G}$ action on E, defined by $\tilde{\sigma}\cdot(\lambda, u) = (\sigma\cdot\lambda, r(\tilde{\sigma})\,u)$, for $(\lambda, u) \in E$ and $\tilde{\sigma} \in \tilde{G}$. If $\{U_\beta, \beta \in \Sigma_n\}$ is a contractible open cover of $\Lambda(X)$, with $\kappa_\beta: U_\beta \to \tilde{G}$, smooth cross-sections over U_β, then a smooth section (frame) over U_β is $s_\beta: \lambda \to (\lambda, r(\kappa_\beta(\lambda))\,\delta_0)$ and transition functions are

$$c_{\beta_1,\beta_2}(\lambda) = \chi(\kappa_{\beta_1}(\lambda)^{-1}\cdot\kappa_{\beta_2}(\lambda)). \tag{5.2}$$

Note $s_{\beta_2}(\lambda) = c_{\beta_1,\beta_2}(\lambda)\,s_{\beta_1}(\lambda)$, for $\lambda \in U_{\beta_1} \cap U_{\beta_2}$. We will now compute the transition functions c_{β_1,β_2} explicitly for a specific choice of U_β, κ_β. Let Σ_n denote the set of all $n\times n$, real symmetric matrices. Let V, V^* be the Lagrangian subspaces introduced earlier (Section 2). If $\Lambda_{\lambda_1}(X) = \{\lambda \in \Lambda(X): \lambda \cap \lambda_1 = 0\}$, then $N_{\lambda_1} = \{\sigma \in SpX: \sigma = id \text{ on } \lambda_1\}$ acts transitively on $\Lambda_{\lambda_1}(X)$. Now $N_V = \{u_\beta = \left(\begin{smallmatrix}I & \beta\\ 0 & I\end{smallmatrix}\right): \beta \in \Sigma_n\}$, $N_{V^*} = \{v_\gamma = \left(\begin{smallmatrix}I & 0\\ \gamma & I\end{smallmatrix}\right): \gamma \in \Sigma_n\}$. Thus $\Lambda_V = \{u_\beta\cdot V^*: \beta \in \Sigma_n\}$. We write $\lambda_\beta = u_\beta\cdot V^*$ and $U_\beta = \Lambda_{\lambda_\beta}(X)$. Then $\{U_\beta: \beta \in \Sigma_n\}$ is an contractible open covering of $\Lambda(X)$. Actually U_β is a coordinate open set and to describe it, note $V \in U_\beta$, since $V \cap \lambda_\beta = (0)$ and N_{λ_β} acts transitively on U_β. Now $N_{\lambda_\beta} = u_\beta N_{V^*} u_\beta^{-1} = \{u_\beta\cdot v_\gamma\cdot u_\beta^{-1}: \gamma \in \Sigma_n\}$. Now $u_\beta v_\gamma\cdot u_\beta^{-1}\cdot V = u_\beta\cdot v_\gamma\cdot V$. Thus $U_\beta = \{u_\beta\cdot v_\gamma\cdot V: \gamma \in \Sigma_n\}$. One checks that the map $\lambda = u_\beta v_\gamma\cdot V \to \gamma$ is one to one on U_β and gives a parametrization of U_β. Define the smooth cross-section $\kappa_\beta: U_\beta \to \tilde{G}$ by

$\lambda = u_\beta \cdot v_\gamma \cdot V \to \tilde{u}_\beta \cdot \tilde{v}_\gamma$ where $\tilde{u}_\beta = (u_\beta, 0) \in \tilde{G}$ and $\tilde{v}_\gamma = \tilde{\tau}\tilde{u}_{-\gamma}\tilde{\tau}^{-1}$. To compute the transition functions we need the following.

Lemma 5.2. *Let $\lambda = u_{\beta_1} \cdot v_{\gamma_1} \cdot V \in U_{\beta_1}$. Then $\lambda \in U_{\beta_2}$ if and only if $\det(I - \gamma_1 \beta) \neq 0$, where $\beta = \beta_2 - \beta_1$. In this case $\lambda = u_{\beta_2} \cdot u_{\gamma_2} \cdot V$, where $\gamma_2 = (I - \gamma_1 \beta)^{-1} \cdot \gamma_1$.*

Proof. Suppose $\lambda = u_{\beta_1} v_{\gamma_1} \cdot V = u_{\beta_2} \cdot v_{\gamma_2} \cdot V$. This is equivalent to $(u_{\beta_1} v_{\gamma_1})^{-1} (u_{\beta_2} v_{\gamma_2}) \in P$ or $v_{-\gamma_1} u_\beta v_{\gamma_2} \in P$ where $\beta = \beta_2 - \beta_1$. Now

$$v_{-\gamma_1} u_\beta v_{\gamma_2} = \begin{pmatrix} I + \beta\gamma_2, & \beta \\ -\gamma_1 + (I - \gamma_1 \beta)\gamma_2, & I - \gamma_1 \beta \end{pmatrix}.$$

$$(I - \gamma_1 \beta)\gamma_2 = \gamma_1 \qquad \text{and} \qquad {}^t(I + \beta\gamma_2) \cdot (I - \gamma_1 \beta) = I$$

and the lemma follows.

Proposition 5.3. *If $\lambda = u_{\beta_1} v_{\gamma_1} \cdot V = u_{\beta_2} \cdot u_{\gamma_2} \cdot V \in U_{\beta_1} \cap U_{\beta_2}$ and $\kappa_{\beta_1}(\lambda)^{-1} \kappa_{\beta_2}(\lambda) = (g, \psi) \in \tilde{P}$, then*

$$\psi = \log |\det(I - \gamma_1 \beta)| + \tfrac{1}{2} i\pi(\operatorname{sgn}(\gamma_2) - \operatorname{sgn}(\gamma_1)).$$

Proof. Now $\kappa_{\beta_1}(\lambda)^{-1} \kappa_{\beta_2}(\lambda) = \tilde{v}_{-\gamma_1} \tilde{u}_\beta \tilde{v}_{\gamma_2} = (g, \psi)$. Now suppose we write $\tilde{v}_\gamma = (v_\gamma, \psi_\gamma) = \tilde{\tau}\tilde{u}_{-\gamma}\tilde{\tau}^{-1}$. Recall $\tilde{\tau} = (\tau, \operatorname{tr}\log)$ and $\tilde{\tau}^{-1} = (\tau^{-1}, -\operatorname{tr}\log \circ \tau^{-1})$. Since $\tau \cdot Z = \tau^{-1} \cdot Z = -Z^{-1}$, we thus have

$$\psi_\gamma = (\operatorname{tr}\log) \cdot u_{-\gamma} \circ \tau^{-1} - (\operatorname{tr}\log) \cdot \tau^{-1}$$

or

$$\psi_\gamma(Z) = \operatorname{tr}\log(-Z^{-1} - \gamma) - \operatorname{tr}\log(-Z^{-1}). \tag{5.3}$$

Next $(g, \psi) = (v_{-\gamma_1}, \psi_{-\gamma_1})(u_\beta, 0)(v_{\gamma_2}, \psi_{\gamma_2})$ and so $\psi = \psi_{-\gamma_1} \circ u_\beta \circ v_{\gamma_2} + \psi_{\gamma_2}$. Or since ψ is a constant, $\psi = \psi \circ v_{-\gamma_2} = \psi_{-\gamma_1} \circ u_\beta + \psi_{\gamma_2} \circ v_{-\gamma_2} = \psi_{-\gamma_1}(Z + \beta) + \psi_{\gamma_2}(Z(-\gamma_2 Z + I)^{-1})$. Substituting for ψ_{γ_1} and ψ_{γ_2} we get

$$\begin{aligned}\psi = {} & \operatorname{tr}\log(-(Z + \beta)^{-1} + \gamma_1) - \operatorname{tr}\log(-(Z + \beta)^{-1}) \\ & + \operatorname{tr}\log(-Z^{-1}) - \operatorname{tr}\log(-Z^{-1} + \gamma_2).\end{aligned} \tag{5.4}$$

Note if γ_1 or γ_2 is $= 0$, then the other one is also 0 from Lemma 5.2. In that case $\psi = 0$. Suppose $\gamma_1 \neq 0$. Then there exists an $\alpha \in O(n)$, such that $\alpha\gamma_1\alpha^{-1} = \left(\begin{smallmatrix} \gamma_1', & 0 \\ 0, & 0 \end{smallmatrix}\right)$ where γ_1' is a $k \times k$, symmetric matrix, with $\det \gamma_1' \neq 0$. If we now write $\alpha\beta\alpha^{-1} = \left(\begin{smallmatrix} \beta', & * \\ *, & * \end{smallmatrix}\right)$, then

$$\alpha(I - \gamma_1 \beta)\alpha^{-1} = \begin{pmatrix} I_k - \gamma_1'\beta', & * \\ 0, & I_{n-k} \end{pmatrix}.$$

If you now define $\gamma_2' = (I_k - \gamma_1'\beta')^{-1}\gamma_1'$, then one checks, $\alpha\gamma_2\alpha^{-1} = \begin{pmatrix}\gamma_2' & 0 \\ 0 & 0\end{pmatrix}$. Moreover $\det\gamma_2' \neq 0$ and $\det(I-\gamma_1\beta) = \det(I_k - \gamma_1'\beta') = (\det\gamma_1')(\det\gamma_2')^{-1}$. We now use Proposition A.2 of the Appendix to simplify (5.4). Note $\alpha(Z+\beta)\alpha^{-1} = \begin{pmatrix} Z'+\beta' & * \\ * & * \end{pmatrix}$. Thus

$$\begin{aligned}\psi &= \operatorname{tr}\log((Z+\beta)' - \gamma_1'^{-1}) + \log|\det\gamma_1'| - i\pi n^-(-\gamma_1)\\ &\quad - \{\operatorname{tr}\log(Z' - \gamma_2'^{-1}) + \log|\det\gamma_2'| - i\pi n^-(-\gamma_2)\}.\end{aligned}$$

Now $(\gamma_2')^{-1} = (\gamma_1')^{-1}(I_k - \gamma_1'\beta') = (\gamma_1')^{-1} - \beta'$. Also $(Z+\beta)' = Z' + \beta'$. Thus

$$\psi = \log|\det\gamma_1'| - \log|\det\gamma_2'| + i\pi(n^-(-\gamma_2) - n^-(-\gamma_1)).$$

Now $n^-(-\gamma_2) - n^-(-\gamma_1) = n^+(\gamma_2) - n^+(\gamma_1) = (1/2)(\operatorname{sgn}\gamma_2 - \operatorname{sgn}\gamma_1)$. ∎

COROLLARY 5.4. *With the same notation as in the Proposition above,*

$$c_{\beta_1,\beta_2}(\lambda) = |\det(g\,|\,V)|^{1/2}\, e^{i\pi(\operatorname{sgn}\gamma_1 - \operatorname{sgn}\gamma_2)/4}.$$

Proof. Since $g = \begin{pmatrix} * & * \\ 0 & 1-\gamma_1\beta\end{pmatrix}$ (see Lemma 5.2), $\det(g|V) = (\det(I_1 - \gamma_1\beta))^{-1}$, the corollary follows from (5.1), (5.2), and the above proposition.

Remark 5.5. Consider the line bundle $\tilde G \times_{|\chi|} \mathbb{C}$. Here $|\chi|(\tilde g) = \exp(-(1/2)\operatorname{Re}\psi) = |\det(g\,|\,V)|^{1/2}$ if $\tilde g = (g, \psi)$. Then the line bundle $\tilde G \times_{|\chi|} \mathbb{C}$ may be identified with the half-density bundle $D^{1/2}$ on λ, i.e., the fiber at $\lambda(\lambda \in \Lambda(X))$, $(D^{1/2})_\lambda =$ set of translation invariant half densities on λ. For example, the translation invariant half densities on V is a one dimensional vector space over $\mathbb{C}$ with the generator denoted by $|dp_1 \wedge \cdots \wedge dp_n|^{1/2} = |dV|^{1/2}$, $p_1, ..., p_n$, being coordinates on V. The line bundle equivalence between $\tilde G \times_{|\chi|} \mathbb{C}$ and $D^{1/2}$ is defined (as for E) by $(\tilde\sigma, z)\tilde P \to (\sigma\cdot V, z(\sigma^{-1})^*\,|dV|^{1/2}) \in D^{1/2} = \{(\lambda, d) : \lambda \in \Gamma(X), d \in D^{1/2}_\lambda\}$. Note $(\sigma^{-1})^*$ is the pull back operation. Note the $\tilde G$ action on $D^{1/2}$ is the natural one, $\tilde\sigma\cdot(\lambda, d) = (\sigma\lambda, (\sigma^{-1})^* d)$, d standing for a half density on λ. Similarly let $\chi_0 = \chi/|\chi|$ where the quasi character χ of $\tilde P$ is (as before) defined by (5.1), and define

$$M = \tilde G \times_{\chi_0} \mathbb{C}. \tag{5.5}$$

It is clear that M has transition functions

$$c^M_{\beta_1,\beta_2}(\lambda) = \chi_0(\kappa_{\beta_1}^{-1}(\lambda)\,\kappa_{\beta_2}(\lambda)) = \exp\frac{i\pi}{4}(\operatorname{sgn}\gamma_1 - \operatorname{sgn}\gamma_2). \tag{5.6}$$

Clearly as homogeneous line bundles, $E = D^{1/2} \otimes M$. Note all the line bundles E, $D^{1/2}$, and M are all equivalent to the trivial line bundle, in view of Lemma 5.1. We will presently identify M as the Maslov bundle, as formally defined by Hormander (see [Hor, Vol. III, p. 334]. In this connection note

$D^{1/2} \otimes D^{1/2}$ is a bundle of translation-invariant densities on λ, $\lambda \in \Lambda(X)$, while $E \otimes E$ is the bundle of (translation-invariant) volume forms on λ, $\lambda \in \Lambda(X)$. Thus E may be considered as the bundle of (translation-invariant) half-forms on $\Lambda(X)$. (In this connection see [B, G-S, Chap. 5].)

We now recall Hormander's definition of the Maslov bundle. Let $\lambda_0, \lambda, \lambda_1, \lambda_2$ in $\Lambda(X)$, be such that λ_1 and λ_2 are both transversal to λ_0 and λ. Then Hormander defines (see [Hor, Vol. III, p. 334])

$$\sigma(\lambda_0, \lambda; \lambda_2, \lambda_1) = \frac{1}{2} \operatorname{sgn} \begin{pmatrix} -A & I \\ I & -B \end{pmatrix}, \tag{5.7}$$

where A and B are symmetric matrices arising in the defining equations of λ_2 and λ; i.e., let x, ξ be symplectic coordinates on X such that

$$\lambda_0 = \{x = 0\}, \qquad \lambda_1 = \{\xi = 0\}, \qquad \lambda_2 = \{\xi = Ax\}, \qquad \lambda = \{x = B\xi\}.$$

The Maslov bundle is defined as the bundle corresponding to the local data $\{\Lambda_{\lambda_1}(X), g_{\lambda_1, \lambda_2}, \lambda_1, \lambda_2 \in \Lambda_{\lambda_0}(X)\}$ where

$$g_{\lambda_1, \lambda_2}(\lambda) = \exp \left\{ \frac{i\pi}{2} \sigma(\lambda_0, \lambda; \lambda_1, \lambda_2) \right\}$$

for $\lambda \in \Lambda_{\lambda_1}(X) \cap \Lambda_{\lambda_2}(X)$. To show that this gives the same bundle as M above, we choose $\lambda_0 = V$ and work with coordinates p, q associated to our fixed symplectic basis (see Section 2), so that $V = \{q = 0\}$ and $V^* = \{p = 0\}$. Let $\lambda_1 = \lambda_{\beta_1} = u_{\beta_1} \cdot V^*$, $\lambda_{\beta_2} = u_{\beta_2} \cdot V^*$, and $\lambda = u_{\beta_1} v_{\gamma_1} \cdot V$, see notation introduced earlier in this section. Then the defining equations are $V = \{q = 0\}$, $\lambda_1 = \{p = \beta_1 q\}$, $\lambda_2 = \{p = \beta_2 q\}$, and $\lambda = \{u_{\beta_1} \cdot v_{\gamma_1} \cdot z : z \in V\} = \{p = (I + \beta_1 \gamma_1) z, q = \gamma_1 z \text{ for some } z \in V\}$. If now we let $x = q$ and $\xi = p - \beta_1 q$, then x, ξ are symplectic coordinates on X in the sense of Hormander and defining equations become $\lambda_0 = V = \{x = 0\}$, $\lambda_1 = \lambda_{\beta_1} = \{\xi = 0\}$, $\lambda_2 = \lambda_{\beta_2} = \{\xi = (\beta_2 - \beta_1) x\}$, and $\lambda = \{x = \gamma_1 \xi\}$. Thus the A and B arising in (5.7) are $A = \beta_2 - \beta_1 = \beta$ (earlier notation see Lemma 5.2) and $B = \gamma_1$.

LEMMA 5.6. *With the above notation* $\operatorname{sgn}\left(\begin{smallmatrix} -A, & I \\ I, & -B \end{smallmatrix}\right) = \operatorname{sgn}(\gamma_2) - \operatorname{sgn}(\gamma_1)$.

Proof. Choose an $\alpha \in O(n)$, so that $\alpha \gamma_1 \alpha^{-1} = \left(\begin{smallmatrix} \gamma_1' & 0 \\ 0 & 0 \end{smallmatrix}\right)$ with $\det \gamma_1' \neq 0$. Write $\alpha \beta \alpha^{-1} = \left(\begin{smallmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{smallmatrix}\right)$. Here γ_1' and β_{11} are $k \times k$ matrices. Let $F = \left(\begin{smallmatrix} -A, & I \\ I, & -B \end{smallmatrix}\right)$ and let

$$H = \begin{pmatrix} I_k, & 0, & \gamma_1'^{-1} & \beta_{12} \\ 0, & 0, & 0, & I_{n-k} \\ 0, & 0, & I_k, & 0 \\ 0, & I_{n-k}, & 0, & \frac{1}{2}\beta_{22}. \end{pmatrix}.$$

Then one checks that

$$HF({}^tH)=\begin{pmatrix} -\beta_{11}+(\gamma_1')^{-1}, & 0, & 0, & 0 \\ 0, & 0, & 0, & I_{n-k} \\ 0, & 0, & -\gamma_1', & 0 \\ 0, & I_{n-k}, & 0, & 0. \end{pmatrix}.$$

Thus

$$\operatorname{sgn} F=\operatorname{sgn}\begin{pmatrix} -\beta_{11}+(\gamma_1')^{-1}, & 0 \\ 0, & -\gamma_1' \end{pmatrix}+\operatorname{sgn}\begin{pmatrix} 0 & I_{n-k} \\ I_{n-k}, & 0 \end{pmatrix}$$

$$=\operatorname{sgn}(-\beta_{11}+(\gamma_1')^{-1})-\operatorname{sgn}(\gamma_1').$$

Now $-\beta_{11}+(\gamma_1')^{-1}=(\gamma_2'^{-1})$ (see the proof of Proposition 5.3). Thus $\operatorname{sgn} F=\operatorname{sgn}(\gamma_2')-\operatorname{sgn}(\gamma_1')=\operatorname{sgn}\gamma_2-\operatorname{sgn}\gamma_1$. ∎

COROLLARY 5.7. *The line bundle M is equivalent to Hormander's Maslov bundle.*

Finally we remark that $E=D^{1/2}\otimes M$ also implies that $M=E\otimes(D^{+1/2})^*$, where $(D^{1/2})^*$ is the dual line bundle of $D^{1/2}$. This dual bundle may be identified with the bundle of half densities on λ^*, $\lambda\in\Lambda(X)$. This identification gives rise to a description of the bundle M as given in Hormander [Hor, p. 332]. In connection with the material of this section see also [Duis, G-S].

APPENDIX: ON THE tr-log FUNCTION

PROPOSITION A.1. *Let $\mathscr{H}_n$ denote the generalised upper half-plane consisting of all $n\times n$, complex symmetric matrices Z, with* $\operatorname{Im} Z>0$. *Then there exists a unique continuous function denoted by* tr log *on $\mathscr{H}_n$ with the following properties*

(i) $\exp \operatorname{tr}\log Z=\det Z$;

(ii) $\operatorname{tr}\log(iI_n)=in\pi/2$; *moreover this function is holomorphic and satisfies*

(iii) $\operatorname{tr}\log(\alpha Z{}^t\alpha)=\operatorname{tr}\log Z+\log|\det\alpha|^2$ *for all* $\alpha\in GL(n,R)$;

(iv) *if* $Z=\operatorname{diag}(z_1,\dots,z_n)$, *with* $z_j\in\mathbb{C}$, $\operatorname{Im} z_j>0$, *then* $\operatorname{tr}\log Z=\Sigma\log z_j$.

Here $\log z$ *is the principal logarithm of the complex number* z.

Proof. Consider the map $\varphi\colon GL(n,R)\times R^n\to\mathscr{H}_n$ defined $\alpha,(a_1,\dots,a_n)\to \alpha\operatorname{diag}(a_1+i,\dots,a_n+i)\cdot{}^t\alpha$. Then this map is surjective. Moreover if $\alpha\operatorname{diag}(a_1+i,\dots,a_n+i)^t\alpha=\beta\operatorname{diag}(b_1+i,\dots,b_n+i)\cdot{}^t\beta$, then it follows that $\beta^{-1}\alpha\in O(n)$ and $a_1,\dots,a_n$ is a permutation of $b_1,\dots,b_n$. Thus the function $H(\alpha, a_1, \ldots, a_n) = \Sigma \log(a_j + i) + \log|\det \alpha|^2$ is a continuous function which is constant on the fibers. From this one deduces easily that there is a continuous function h on $\mathscr{H}_n$ such that $H(\alpha, a_1, \ldots, a_n) = h(\alpha \cdot \mathrm{diag}(a_1 + i, \ldots, a_n + i) \cdot {}^t\alpha)$. One checks easily that the function $h(Z)$ on $\mathscr{H}_n$ has the stated properties. We will write $h(Z)$ as $\operatorname{tr}\log Z$ the trace of the (principal) logarithm of Z.

Remark. Actually this function can be exhibited as the trace of a matrix, which can be considered as the logarithm of Z. We do not go into this, as this is not needed. We will however state a precise result (see [V, p. 111]). Let $\Omega = \{g \in GL(n, \mathbb{C})\colon \operatorname{spec} g \subset \mathbb{C} \setminus (-\infty, 0]\}$, $\omega = \{A \in gl(n, \mathbb{C})\colon \operatorname{spec} A \subset (\lambda \in \mathbb{C}\colon |\operatorname{Im} \lambda| < \pi)\}$. Then ω, Ω are open, and the exponential map is an analytic diffeomorphism of ω onto Ω and the inverse map $\log\colon \Omega \to \omega$ is called the principal logarithm. Note when $Z \in \mathscr{H}_n$, Z has no real eigenvalues and in fact one checks that the function $\operatorname{tr}\log Z$, introduced above coincides with the trace of $\log Z$, for $Z \in \mathscr{H}_n$. An analytic formula for log is given in [Sou-2].

PROPOSITION A.2 *For a real symmetric matrix* ρ,

$$\begin{aligned}\operatorname{tr}\log(-Z^{-1} - \rho) - \operatorname{tr}\log(-Z^{-1}) - \operatorname{tr}\log(Z' + \rho'^{-1})\\ = \log|\det \rho'| - i\pi n_-(\rho),\end{aligned}$$

where $n_-(\rho)$ *is the number of negative eigenvalues of* ρ *and matrices* ρ' *and* Z' *are defined as follows. Let* $\alpha \in O(n)$ *be such that* $\alpha\rho\alpha^{-1} = \begin{pmatrix}\rho' & 0\\ 0 & 0\end{pmatrix}$, *with* $\det \rho' \neq 0$; *and let* Z' *be defined by* $\alpha Z \alpha^{-1} = \begin{pmatrix}Z' & *\\ * & *\end{pmatrix}$. *In particular when* $\det \rho \neq 0$, *we have the identity*

$$\begin{aligned}\operatorname{tr}\log(-Z^{-1} - \rho) - \operatorname{tr}\log(-Z^{-1})\\ = \operatorname{tr}\log(Z + \rho^{-1}) + \log|\det \rho| - i\pi n_-(\rho).\end{aligned}$$

Proof. Let $\psi(Z)$ denote the left-hand side, then $\exp \psi(Z) = \det \rho'$, as is easily checked. Thus $\psi(Z)$ is a constant. So it is sufficient to evaluate it when $Z = iI_n$. Because of property (3) in Proposition A.1 of tr log, we may assume that ρ is diagonal $= \mathrm{diag}(\rho', 0)$. Then $Z' = iI_k$. Thus

$$\begin{aligned}\psi(iI_n) &= \operatorname{tr}\log(iI_n - \rho) - \operatorname{tr}\log(iI_n) - \operatorname{tr}\log(iI + \rho'^{-1})\\ &= \operatorname{tr}\log(iI_k - \rho') - \operatorname{tr}\log iI_k - \operatorname{tr}\log(iI_k + \rho'^{-1})\\ &= \sum_{j=1}^{k} \{\log(i - a_j) - \log i - \log(i + a_j^{-1})\}.\end{aligned}$$

Now

$$\log(i - a_j) - \log i - \log(i + a_j^{-1}) = \log|a_j| - i\pi n_-(a_j),$$

where $n\ \ (a_j)=0$ if $a_j>0$ and $=1$ if $a_j<0$. Thus

$$\psi(iI_n)=\log|\det\rho'|-i\pi n\ \ (\rho).$$

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