APPROXIMATE FORMULAS FOR SOME FUNCTIONS OF PRIME NUMBERS

Dedicated to Hans Rademacher on the occasion of his seventieth birthday

BY

J. Barkley Rosser and Lowell Schoenfeld

1. Acknowledgments

The major portion of Rosser's work on this paper was done at Cornell University with support from the Air Force Office of Scientific Research. His recent work was done as an employee of the Communications Research Division (Project FOCUS) of the Institute for Defense Analyses. Schoenfeld's work was begun at the Westinghouse Research Laboratories and was completed at the Pennsylvania State University.

Three major computations were performed to obtain various of the data listed herein. The first was performed some nine years ago by workers under the direction of Mr. Marvin Howard at the Institute for Numerical Analysis, then part of the National Bureau of Standards, with support from the Office of Naval Research. The second was performed some four years ago with the cooperation of Professor R. J. Walker on the IBM 650 at the Cornell Computing Center, with support from the National Science Foundation. The third was performed recently with the cooperation of Dr. Kenneth I. Appel at the Computing Center of and with the support of Project FOCUS.

2. Introduction

Counting 2 as the first prime, we denote by $\pi(x)$, $\vartheta(x)$, and $\psi(x)$, respectively, the number of primes $\leqq x$, the logarithm of the product of all primes $\leqq x$, and the logarithm of the least common multiple of all positive integers $\leqq x$; if $x < 2$, we take $\pi(x) = \vartheta(x) = \psi(x) = 0$. We also let p_n denote the n^{th} prime, and $\phi(n)$ denote the number of positive integers $\leqq n$ and relatively prime to n. Throughout, n shall denote a positive integer, p a prime, and x a real number. We shall present approximate formulas for $\pi(x)$, $\vartheta(x)$, $\psi(x)$, p_n, $\phi(n)$, and other functions related to prime numbers.

In 1808, on the basis of attempting to fit known values of $\pi(x)$ by an empirical formula, Legendre conjectured an approximation very similar to that given below in (2.19). In 1849, again on the basis of counts of the number of primes in various intervals, Gauss communicated to Encke a conjecture that in the neighborhood of the number x the average density of the primes is $1/\log x$. On this basis, if one should wish an estimate for the sum of $f(p)$ over all primes $p \leqq x$, the natural approximation would be

Received February 13, 1961.

$$\sum_{p \leqq x} f(p) \cong \int_2^x \frac{f(y)\,dy}{\log y}. \tag{2.1}$$

Consequently, one would presume the following approximations:

$$\pi(x) = \sum_{p \leqq x} 1 \cong \int_2^x \frac{dy}{\log y}, \tag{2.2}$$

$$\vartheta(x) = \sum_{p \leqq x} \log p \cong \int_2^x dy \cong x, \tag{2.3}$$

$$\sum_{p \leqq x} \frac{1}{p} \cong \int_2^x \frac{dy}{y \log y} \cong \log\log x + B, \tag{2.4}$$

$$\sum_{p \leqq x} \frac{\log p}{p} \cong \int_2^x \frac{dy}{y} \cong \log x + E, \tag{2.5}$$

$$\begin{aligned} \prod_{\alpha < p \leqq x} \left(1 - \frac{\alpha}{p}\right) &= \exp\left\{\sum_{\alpha < p \leqq x} \log\left(1 - \frac{\alpha}{p}\right)\right\} \\ &\cong \exp\left\{c_1(\alpha) - \alpha \sum_{p \leqq x} \frac{1}{p}\right\} \cong \frac{c(\alpha)}{(\log x)^\alpha}, \end{aligned} \tag{2.6}$$

where α is a real constant, usually taken to be unity.

In (2.4) we have indicated a "constant of integration," B, whose value is taken to be

$$\lim_{x\to\infty}\{\textstyle\sum_{p \leqq x} 1/p - \log\log x\}.$$

Because this limit exists, the absolute error in (2.4) tends to zero as x tends to infinity. In (2.5) and (2.6), we have indicated constants E, $c_1(\alpha)$, and $c(\alpha)$ for analogous reasons. In (2.2) and (2.3), no constants are indicated because the limits to which they would correspond do not exist.

The validity of the approximations (2.2) through (2.6) was rigorously established just before the turn of the century by Hadamard and de la Vallée Poussin. A very excellent account of these matters is given in Ingham [6], together with extensive references to the literature.

From Ingham [6], we can get alternate expressions for B, E, and $c(1)$ as follows:

$$B = C + \textstyle\sum_p \{\log[1 - (1/p)] + (1/p)\}, \tag{2.7}$$

$$E = -C - \textstyle\sum_{n=2}^{\infty} \sum_p (\log p)/p^n, \tag{2.8}$$

$$c(1) = e^{-C}, \tag{2.9}$$

where C is Euler's constant. We find (2.7) and (2.9) on pp. 22–23 of Ingham [6], and can derive (2.8) from a formula near the top of p. 81. Approximate numerical values are:

$$B = 0.26149\,72128\,47643, \tag{2.10}$$

$$E = -1.33258\,22757\,33221, \tag{2.11}$$

(2.12) $$c(1) = 0.56145\,94835\,66885,$$

(2.13) $$1/c(1) = 1.78107\,24179\,90198,$$

(2.14) $$c(2) = 0.83242\,90656\,62.$$

Let us define the logarithmic integral $\mathrm{li}(x)$ by

(2.15) $$\mathrm{li}(x) = \mathrm{Ei}(\log x),$$

where $\mathrm{Ei}(y)$ is the exponential integral, defined by

(2.16) $$\mathrm{Ei}(y) = \lim_{\varepsilon \to 0+} \left\{ \int_{-\infty}^{-\varepsilon} \frac{e^t\,dt}{t} + \int_{\varepsilon}^{y} \frac{e^t\,dt}{t} \right\}.$$

Then

(2.17) $$\int_2^x \frac{dy}{\log y} = \mathrm{li}(x) - \mathrm{li}(2).$$

Consequently, in place of (2.2), it is common to use $\mathrm{li}(x)$ as an approximation for $\pi(x)$. This is more convenient than (2.2) because $\mathrm{Ei}(y)$ has been extensively tabulated; a convenient tabulation is given in the W.P.A. Tables [18], in which the reader should note the supplementary Table III at the end of Vol. II.

If we use two terms of the asymptotic expansion for $\mathrm{li}(x)$, we get the following convenient approximation:

(2.18) $$\pi(x) \cong \frac{x}{\log x}\left(1 + \frac{1}{\log x}\right).$$

Using a closer approximation for $\mathrm{li}(x)$ gives the sharper result:

(2.19) $$\pi(x) \cong \frac{x}{\log x - 1}.$$

From this, we see that a suitable approximation for p_n is given by

(2.20) $$p_n \cong n(\log n + \log\log n - 1).$$

Much work has been done in estimating the orders of magnitude of the errors in the various approximations listed above. A classic result appears as Theorem 23 on p. 65 of Ingham [6] in the form

(2.21) $$\pi(x) = \mathrm{li}(x) + O(x \exp\{-a(\log x)^{1/2}\}),$$

where a is a positive absolute constant. This means that there are positive absolute constants a, b, and X such that for $x \geqq X$

(2.22) $$|\,\pi(x) - \mathrm{li}(x)\,| < bx \exp\{-a(\log x)^{1/2}\}.$$

Improvements of this sort of result continue to appear. The sharpest known, given in Vinogradov [17], is

(2.23) $$\pi(x) = \mathrm{li}(x) + O(x \exp\{-a(\log x)^{3/5}\}).$$

Undoubtedly this is not the best possible result, but the precise behavior of $\pi(x) - \mathrm{li}(x)$ depends on the location of the zeros of the Riemann zeta function, and cannot be determined until we have more precise information about them than we have now. A discussion of this point appears in Chap. IV of Ingham [6].

Even in our present ignorance about the zeros of the zeta function, it can be shown that $\mathrm{li}(x)$ alone cannot be a wholly satisfactory approximation to $\pi(x)$. Specifically, it has been shown that there is a sequence of values of x, tending to infinity, at which alternately

$$\pi(x) - \mathrm{li}(x) > x^{1/2}/\log x$$

and

$$\pi(x) - \mathrm{li}(x) < - x^{1/2}/\log x.$$

Indeed, Theorem 35 on p. 103 of Ingham [6] states a significantly stronger result. An analogous result for $\vartheta(x) - x$ follows from Theorem 34 on p. 100 of Ingham [6] by means of the relations

$$\psi(x) = \sum_{n=1}^{\infty} \vartheta(x^{1/n}), \tag{2.24}$$

$$\vartheta(x) = \sum_{n=1}^{\infty} \mu(n)\psi(x^{1/n}). \tag{2.25}$$

Each of these summations is in fact only finite, since the summands become zero as soon as $n > (\log x)/(\log 2)$. The first of these equations is derived on p. 12 of Ingham [6], and inversion of the first gives the second, in which μ is the Möbius function defined on p. 567 of Landau [7], vol. 2.

Once one has a good estimate for $\pi(x) - \mathrm{li}(x)$, one can get an approximation for sums of functions of primes as follows. Using the Stieltjes integral, one has

$$\sum_{p \leqq x} f(p) = \int_{2-}^{x} f(y)\, d\pi(y).$$

Integration by parts gives

$$\begin{aligned}\sum_{p \leqq x} f(p) &= f(x)\pi(x) - \int_2^x f'(y)\pi(y)\, dy \\ &= f(x)\pi(x) - \int_2^x f'(y)\, \mathrm{li}(y)\, dy - \int_2^x f'(y)\{\pi(y) - \mathrm{li}(y)\}\, dy.\end{aligned}$$

Then integration by parts gives

$$\begin{aligned}\sum_{p \leqq x} f(p) = \int_2^x \frac{f(y)\, dy}{\log y} &+ f(2)\, \mathrm{li}(2) \\ &+ f(x)\{\pi(x) - \mathrm{li}(x)\} - \int_2^x f'(y)\{\pi(y) - \mathrm{li}(y)\}\, dy,\end{aligned} \tag{2.26}$$

which is the precise version of (2.1). If the integral in (2.28) below converges, we can rewrite (2.26) as

$$\sum_{p \leqq x} f(p) = \int_2^x \frac{f(y)\,dy}{\log y} + K_f \tag{2.27}$$
$$+ f(x)\{\pi(x) - \mathrm{li}(x)\} + \int_x^\infty f'(y)\{\pi(y) - \mathrm{li}(y)\}\,dy,$$

where K_f is the constant given by

$$K_f = f(2)\,\mathrm{li}(2) - \int_2^\infty f'(y)\{\pi(y) - \mathrm{li}(y)\}\,dy. \tag{2.28}$$

Using these, we can get sharper forms of (2.3) through (2.6). Thus, from (2.21) and (2.26), we get

$$\vartheta(x) = x + O(x \exp\{-a(\log x)^{1/2}\}). \tag{2.29}$$

From (2.21) and (2.27), we get

$$\textstyle\sum_{p \leqq x} 1/p = \log\log x + B + O(\exp\{-a(\log x)^{1/2}\}), \tag{2.30}$$

$$\textstyle\sum_{p \leqq x} (\log p)/p = \log x + E + O(\exp\{-a(\log x)^{1/2}\}). \tag{2.31}$$

From (2.30) we proceed as in (2.6) to get

$$\prod_{\alpha < p \leqq x} \left(1 - \frac{\alpha}{p}\right) = \frac{c(\alpha)}{(\log x)^\alpha} + O(\exp\{-a(\log x)^{1/2}\}). \tag{2.32}$$

From (2.29) and (2.24), one can get a formula for $\psi(x)$ analogous to (2.29). By starting from (2.23) rather than (2.21), one can get even sharper results than (2.29) through (2.32).

Though results like those above are interesting, and are difficult to prove, they are of little use for getting dependable numerical approximations unless values of a, b, and X in (2.22) are furnished; this is seldom done. In Rosser [12], explicit bounds were presented for the errors in our approximations. More recently, much better bounds have been obtained by using modern computing machinery and taking advantage of new information about the zeros of the zeta function. These results will be stated in the early part of the present paper, with the proofs being mainly withheld until the later sections.

3. Widely applicable approximations

For a very sharp approximation, one must either use complicated formulas or be satisfied with validity over a limited range. In this section, we shall list approximations which combine the advantages of being reasonably simple, reasonably precise, and valid for nearly all values. Note that Theorem 1 below will replace (2.18) by closely related and specific inequalities, while Theorems 2–7 will do the same for (2.19), (2.20), (2.3), (2.4), (2.5), and (2.6) respectively. Theorem 8 is a variant of Theorem 7 which is sometimes more convenient.

THEOREM 1. *We have*

$$\frac{x}{\log x}\left(1+\frac{1}{2\log x}\right)<\pi(x) \qquad \textit{for } 59 \leqq x, \tag{3.1}$$

$$\pi(x)<\frac{x}{\log x}\left(1+\frac{3}{2\log x}\right) \qquad \textit{for } 1<x. \tag{3.2}$$

THEOREM 2. *We have*

$$x/(\log x-\tfrac{1}{2})<\pi(x) \qquad \textit{for } 67 \leqq x, \tag{3.3}$$

$$\pi(x)<x/(\log x-\tfrac{3}{2}) \qquad \textit{for } e^{3/2}<x \tag{3.4}$$

(*and hence for* $4.48169 \leqq x$).

COROLLARY 1. *We have*

$$x/\log x<\pi(x) \qquad \textit{for } 17 \leqq x, \tag{3.5}$$

$$\pi(x)<1.25506\, x/\log x \qquad \textit{for } 1<x. \tag{3.6}$$

COROLLARY 2. *For* $1<x<113$ *and for* $113.6 \leqq x$

$$\pi(x)<5x/(4\log x). \tag{3.7}$$

COROLLARY 3. *We have*

$$3x/(5\log x)<\pi(2x)-\pi(x) \qquad \textit{for } 20\tfrac{1}{2} \leqq x, \tag{3.8}$$

$$0<\pi(2x)-\pi(x)<7x/(5\log x) \qquad \textit{for } 1<x. \tag{3.9}$$

For the ranges of x for which these corollaries do not follow directly from the theorem, they can be verified by reference to Lehmer's table of primes [10]. A similar remark applies to all corollaries of this section unless a proof is indicated.

The inequality (3.8) improves a result of Finsler [3]. The left side of (3.9) is just the classic result, conjectured by Bertrand (and known as Bertrand's Postulate) and proved in Tchebichef [14], that there is at least one prime between x and $2x$. The right side of (3.9) gives a result of Finsler [3], with Finsler's integral n replaced by our real x. Finsler's elementary proofs are reproduced in Trost [15] on p. 58. The relation (3.12) below states a result of Rosser [11].

THEOREM 3. *We have*

$$n(\log n+\log\log n-\tfrac{3}{2})<p_n \qquad \textit{for } 2 \leqq n, \tag{3.10}$$

$$p_n<n(\log n+\log\log n-\tfrac{1}{2}) \qquad \textit{for } 20 \leqq n. \tag{3.11}$$

COROLLARY. *We have*

$$n\log n<p_n \qquad \textit{for } 1 \leqq n, \tag{3.12}$$

$$p_n<n(\log n+\log\log n) \qquad \textit{for } 6 \leqq n. \tag{3.13}$$

THEOREM 4. *We have*

$$x(1 - 1/(2\log x)) < \vartheta(x) \qquad \text{for } 563 \leqq x, \tag{3.14}$$

$$\vartheta(x) < x(1 + 1/(2\log x)) \qquad \text{for } 1 < x. \tag{3.15}$$

COROLLARY. *We have*

$$x(1 - 1/\log x) < \vartheta(x) \qquad \text{for } 41 \leqq x. \tag{3.16}$$

THEOREM 5. *We have*

$$\log\log x + B - 1/(2\log^2 x) < \textstyle\sum_{p \leqq x} 1/p \qquad \text{for } 1 < x, \tag{3.17}$$

$$\textstyle\sum_{p \leqq x} 1/p < \log\log x + B + 1/(2\log^2 x) \qquad \text{for } 286 \leqq x. \tag{3.18}$$

COROLLARY. *We have*

$$\log\log x < \textstyle\sum_{p \leqq x} 1/p \qquad \text{for } 1 < x, \tag{3.19}$$

$$\textstyle\sum_{p \leqq x} 1/p < \log\log x + B + 1/\log^2 x \qquad \text{for } 1 < x. \tag{3.20}$$

THEOREM 6. *We have*

$$\log x + E - 1/(2\log x) < \textstyle\sum_{p \leqq x} (\log p)/p \qquad \text{for } 1 < x, \tag{3.21}$$

$$\textstyle\sum_{p \leqq x} (\log p)/p < \log x + E + 1/(2\log x) \qquad \text{for } 319 \leqq x. \tag{3.22}$$

COROLLARY. *We have*

$$\textstyle\sum_{p \leqq x} (\log p)/p < \log x + E + 1/\log x \qquad \text{for } 32 \leqq x, \tag{3.23}$$

$$\textstyle\sum_{p \leqq x} (\log p)/p < \log x \qquad \text{for } 1 < x. \tag{3.24}$$

THEOREM 7. *We have*

$$\frac{e^{-C}}{\log x}\left(1 - \frac{1}{2\log^2 x}\right) < \prod_{p \leqq x}\left(1 - \frac{1}{p}\right) \qquad \text{for } 285 \leqq x, \tag{3.25}$$

$$\prod_{p \leqq x}\left(1 - \frac{1}{p}\right) < \frac{e^{-C}}{\log x}\left(1 + \frac{1}{2\log^2 x}\right) \qquad \text{for } 1 < x. \tag{3.26}$$

COROLLARY. *We have*

$$\frac{e^{-C}}{\log x}\left(1 - \frac{1}{\log^2 x}\right) < \prod_{p \leqq x}\left(1 - \frac{1}{p}\right) \qquad \text{for } 1 < x. \tag{3.27}$$

THEOREM 8. *We have*

$$e^{C}(\log x)\left(1 - \frac{1}{2\log^2 x}\right) < \prod_{p \leqq x} \frac{p}{p-1} \qquad \text{for } 1 < x, \tag{3.28}$$

$$\prod_{p \leqq x} \frac{p}{p-1} < e^{C}(\log x)\left(1 + \frac{1}{2\log^2 x}\right) \qquad \text{for } 286 \leqq x. \tag{3.29}$$

COROLLARY 1. *We have*

$$\prod_{p \leqq x} \frac{p}{p-1} < e^{C}(\log x)\left(1 + \frac{1}{\log^2 x}\right) \qquad \text{for } 1 < x. \tag{3.30}$$

COROLLARY 2. *We have*

$$\prod_{p \leqq x} \frac{p}{p-1} < e^{C} \sum_{1 \leqq n \leqq x} \frac{1}{n} \qquad \textit{for } 1 \leqq x. \tag{3.31}$$

THEOREM 9. *We have*

$$\vartheta(x) < 1.01624\, x \qquad \textit{for } 0 < x. \tag{3.32}$$

For a better bound for $\vartheta(x)$ when $x \leqq 10^8$, note Theorem 18 below.

THEOREM 10. *For $d \leqq x$, we have $cx < \vartheta(x)$ for each of the following pairs of values of c and d:*

c	.980	.975	.970	.965	.960	.955	.950	.945	.94	.93	.92	.91	.89	.86	.84
d	7481	5381	3457	2657	1481	1433	1427	853	809	599	557	349	227	149	101

THEOREM 11. *Let*

$$R = \frac{515}{(\sqrt{546} - \sqrt{322})^2} \quad \textit{and} \quad \varepsilon(x) = (\log x)^{1/2} \exp\{-\sqrt{(\log x)/R}\}.$$

Then we have

$$\{1 - \varepsilon(x)\}\, x < \vartheta(x) \leqq \psi(x) \qquad \textit{for } 2 \leqq x, \tag{3.33}$$

$$\vartheta(x) \leqq \psi(x) < \{1 + \varepsilon(x)\}\, x \qquad \textit{for } 1 \leqq x. \tag{3.34}$$

An approximate value for R is

$$R = 17.51631.$$

THEOREM 12. *The quotient $\psi(x)/x$ takes its maximum at $x = 113$, and*

$$\psi(x) < 1.03883\, x \qquad \textit{for } 0 < x. \tag{3.35}$$

THEOREM 13. *The quotient $\{\psi(x) - \vartheta(x)\}/x^{1/2}$ takes its maximum at $x = 361$, and*

$$\psi(x) - \vartheta(x) < 1.42620\, x^{1/2} \qquad \textit{for } 0 < x. \tag{3.36}$$

THEOREM 14. *We have*

$$0.98\, x^{1/2} < \psi(x) - \vartheta(x) \qquad \textit{for } 121 \leqq x, \tag{3.37}$$

$$\psi(x) - \vartheta(x) < \vartheta(x^{1/2}) + 3x^{1/3} \qquad \textit{for } 0 < x. \tag{3.38}$$

COROLLARY. *We have*

$$\psi(x) - \vartheta(x) < 1.02\, x^{1/2} + 3x^{1/3} \qquad \textit{for } 0 < x. \tag{3.39}$$

Proof. Use Theorem 9.

THEOREM 15. *For $2 \leqq n$*

$$1 + 1/(n-1) \leqq n/\phi(n); \tag{3.40}$$

also for $3 \leqq n$

$$n/\phi(n) < e^C \log\log n + 5/(2 \log\log n) \tag{3.41}$$

except when

$$n = 2230\,92870 = 2\cdot3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot19\cdot23$$

in which case

$$n/\phi(n) < e^C \log\log n + 2.50637/\log\log n. \tag{3.42}$$

In (3.40), equality is attained whenever n is a prime. Thus, by taking n to be a large prime, we can make $n/\phi(n)$ arbitrarily close to unity. It is shown in Landau [7], pp. 217–219, vol. 1, that for each positive ε there are an infinity of n's for which

$$(e^C - \varepsilon) \log\log n < n/\phi(n).$$

We do not know if there are an infinity of n's for which

$$e^C \log\log n \leqq n/\phi(n).$$

4. Special approximations for limited ranges

THEOREM 16. *We have*

$$\mathrm{li}(x) - \mathrm{li}(x^{1/2}) < \pi(x) \qquad \textit{for } 11 \leqq x \leqq 10^8, \tag{4.1}$$

$$\pi(x) < \mathrm{li}(x) \qquad \textit{for } 2 \leqq x \leqq 10^8. \tag{4.2}$$

THEOREM 17. *We have*

$$x - \{\mathrm{li}(x) - \pi(x)\} \log x < \vartheta(x) \qquad \textit{for } e^3 \leqq x \leqq 10^8, \tag{4.3}$$

$$\vartheta(x) < x - 2x^{1/2} + \{\pi(x) - \mathrm{li}(x) + \mathrm{li}(x^{1/2})\} \log x \quad \textit{for } e^4 \leqq x \leqq 10^8. \tag{4.4}$$

By use of Theorem 17 and the Rand table of primes [1], one can get quite sharp estimates of $\vartheta(x)$ for $e^4 \leqq x \leqq 10^8$. However, it is usually adequate to use the more convenient but less precise results below.

THEOREM 18. *For* $0 < x \leqq 10^8$

$$x - 2.05282\, x^{1/2} < \vartheta(x) < x. \tag{4.5}$$

THEOREM 19. *For* $0 < x \leqq 1420.9$ *and for* $1423 \leqq x \leqq 10^8$

$$x - 2\, x^{1/2} < \vartheta(x). \tag{4.6}$$

The coefficient in (4.5) corrects a transposition of digits in Theorem 6 of Rosser [12].

THEOREM 20. *For* $1 < x \leqq 10^8$

$$\log\log x + B < \textstyle\sum_{p\leqq x} 1/p < \log\log x + B + 2/(x^{1/2} \log x). \tag{4.7}$$

THEOREM 21. *For* $0 < x \leqq 10^8$

$$\log x + E < \textstyle\sum_{p\leqq x} (\log p)/p < \log x + E + 2.06123/x^{1/2}. \tag{4.8}$$

THEOREM 22. *For* $0 < x < 113$ *and for* $113.8 \leqq x \leqq 10^8$

$$\sum_{p \leqq x} (\log p)/p < \log x + E + 2/x^{1/2}. \tag{4.9}$$

THEOREM 23. *For* $0 < x \leqq 10^8$

$$e^C \log x < \prod_{p \leqq x} p/(p-1) < e^C \log x + 2e^C/x^{1/2}. \tag{4.10}$$

THEOREM 24. *We have*

$$x^{1/2} < \psi(x) - \vartheta(x) \qquad \textit{for } 121 \leqq x \leqq 10^{16}, \tag{4.11}$$

$$\psi(x) - \vartheta(x) < x^{1/2} + 3x^{1/3} \qquad \textit{for } \quad 0 < x \leqq 10^{16}. \tag{4.12}$$

One immediately wonders if the results of Theorems 20–23 could be valid for all x. It is known that each of (4.1), (4.2), and (4.5) fails infinitely often for large x, and indeed each side of (4.5) fails infinitely often (see Theorems 35 and 34 on pp. 103 and 100 of Ingham [6]). Perhaps one can extend these results to show that each of (4.7) through (4.10) fails for large x; we have not investigated the matter.

Theorem 18 gives sharp bounds for $\vartheta(x)$ for $0 < x \leqq 10^8$. For larger values of x, sharp bounds for $\vartheta(x)$ can be obtained by use of Theorem 14 and its corollary provided sharp bounds for $\psi(x)$ are known. For $10^8 \leqq x \leqq e^{5000}$, sharp bounds for $\psi(x)$ can be obtained from our Table I, in which we tabulate against b values of ε such that for $e^b \leqq x$

$$(1 - \varepsilon)\, x < \psi(x) < (1 + \varepsilon)\, x.$$

The values of m listed pertain to the computations by which Table I was established, all of which will be explained later.

Finally, Theorem 11 can be used to get close approximations to both $\psi(x)$ and $\vartheta(x)$ for large x beyond the range of existing tables. Although Theorem 11 is valid for small values of x as well as large, for x below about e^{3000} it gives poorer estimates for $\vartheta(x)$ than can be obtained from Theorems 4, 9, 10, 18, and 19. From $e^{18.4}$ to e^{4800}, Theorem 11 gives poorer estimates than can be obtained from Table I with the help of Theorem 14.

We can use our sharp estimates for $\vartheta(x)$ to get sharp estimates for other functions depending on primes. Using the Stieltjes integral, we can write

$$\sum_{p \leqq x} f(p) = \int_{2-}^{x} \frac{f(y)}{\log y}\, d\vartheta(y).$$

An integration by parts gives

$$\sum_{p \leqq x} f(p) = \frac{f(x)\vartheta(x)}{\log x} - \int_2^x \vartheta(y)\, \frac{d}{dy}\left(\frac{f(y)}{\log y}\right) dy. \tag{4.13}$$

Alternatively, one can derive (4.13) by use of Theorem A on p. 18 of Ingham

[6]. From (4.13), as in the derivation of (2.26), we get

$$\sum_{p \leqq x} f(p) = \int_2^x \frac{f(y)\,dy}{\log y} + \frac{2f(2)}{\log 2} + \frac{f(x)\{\vartheta(x) - x\}}{\log x} - \int_2^x \{\vartheta(y) - y\} \frac{d}{dy}\left(\frac{f(y)}{\log y}\right) dy. \tag{4.14}$$

For suitable f, we can write (4.14) as

$$\sum_{p \leqq x} f(p) = \int_2^x \frac{f(y)\,dy}{\log y} + L_f + \frac{f(x)\{\vartheta(x) - x\}}{\log x} + \int_x^\infty \{\vartheta(y) - y\} \frac{d}{dy}\left(\frac{f(y)}{\log y}\right) dy, \tag{4.15}$$

where L_f is the constant given by

$$L_f = \frac{2f(2)}{\log 2} - \int_2^\infty \{\vartheta(y) - y\} \frac{d}{dy}\left(\frac{f(y)}{\log y}\right) dy. \tag{4.16}$$

From (4.13) we get

$$\pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(y)\,dy}{y \log^2 y}, \tag{4.17}$$

$$\sum_{p \leqq x} \frac{1}{p} = \frac{\vartheta(x)}{x \log x} + \int_2^x \frac{\vartheta(y)(1 + \log y)}{y^2 \log^2 y}\,dy, \tag{4.18}$$

$$\sum_{p \leqq x} \frac{\log p}{p} = \frac{\vartheta(x)}{x} + \int_2^x \frac{\vartheta(y)\,dy}{y^2}. \tag{4.19}$$

From (4.15) we get

$$\sum_{p \leqq x} \frac{1}{p} = \log\log x + B + \frac{\vartheta(x) - x}{x \log x} - \int_x^\infty \frac{\{\vartheta(y) - y\}(1 + \log y)}{y^2 \log^2 y}\,dy, \tag{4.20}$$

$$\sum_{p \leqq x} \frac{\log p}{p} = \log x + E + \frac{\vartheta(x) - x}{x} - \int_x^\infty \frac{\vartheta(y) - y}{y^2}\,dy. \tag{4.21}$$

To prove (4.20), it suffices to show that in this case $L_f - \log\log 2 = B$; to show this, we let $x \to \infty$ in (4.15) and use Theorem 11 and the definition of B. The proof of (4.21) is similar.

5. Tabular and computational results, and proofs derived therefrom

The well-known table of primes, Lehmer [10], lists all primes less than 10^7 in such a fashion that one can easily obtain the corresponding value of $\pi(x) + 1$; since Lehmer takes unity to be the first prime, his count of primes differs by unity from ours. Just recently the Rand Corporation has prepared a list of primes, published on microcards in Baker and Gruenberger [1], giving all primes up to slightly beyond 10^8. Within the large range of these tables, one can read off exact values of $\pi(x)$ and p_n. Their count agrees with Lehmer's rather than ours.

By use of Lehmer's Table, Rosser verified Theorem 16 and the right half of (4.5) for $x \leqq 10^6$. An account of his methods is given in Rosser [12]. By the same methods, these results were extended to 10^7 in the first major computation cited in Section 1.

We turn next to the calculations required to establish (2.10) through (2.14). The values for $c(1)$ and $1/c(1)$ given there were computed from (2.9) and the known value of C. The values for B and $c(2)$ given in (2.10) and (2.14) have been taken from pp. 43 and 44 of Rosser [11]. At the end of the footnote on p. 43 of this reference, there is given a twenty-four-decimal value of $C - B$; this value corrects the slightly erroneous value of $B - C$ given in Table I of Gram [5]. Incidentally, Gram reproduces Merrifield's incorrect value of $\sum p^{-3}$ in his Table I; a correct value is given on p. 249 of Davis [2].

To determine the remaining quantity, E, we use the relation

$$- \zeta'(s)/\zeta(s) = \sum_p (\log p)/(p^s - 1) = \sum_p \sum_{r=1}^{\infty} (\log p)/p^{rs}$$

which is given near the bottom of p. 17 of Ingham [6]. From this we obtain for $n > 1$

$$\sum_p (\log p)/p^n = -\sum_{m=1}^{\infty} \mu(m)\, \zeta'(mn)/\zeta(mn). \tag{5.1}$$

Substituting this into (2.8) gives

$$E = - C - \sum_{m=2}^{\infty} \mu(m)\, \zeta'(m)/\zeta(m). \tag{5.2}$$

We first computed $\zeta'(n)$ by using an electronic computer to sum the first 500 terms of the series

$$\zeta'(n) = -\sum_{m=2}^{\infty} (\log m)/m^n;$$

the remainder of the series was computed by the Euler-Maclaurin sum formula. As a check, this was repeated with the first 1000 terms. Using the values of $\zeta(n)$ given on p. 244 of Davis [2], values of

$$- \zeta'(n)/\zeta(n) \quad \text{and} \quad \sum_p (\log p)/p^n$$

were computed for $2 \leqq n \leqq 56$ by using (5.1). These, together with the values of $-\zeta'(n)$ are listed for $2 \leqq n \leqq 29$ in Table IV. In this table the values given for $-\zeta'(n)$ are in error by less than 10^{-17}. Division by $\zeta(n)$ could cause slightly greater errors in $-\zeta'(n)/\zeta(n)$. As we used seventeen-decimal values in (5.1), the errors in $\sum_p p^{-n} \log p$ could be a bit greater. For $n > 29$, the three functions tabulated do not differ in the first seventeen decimals, and each is given to better than seventeen decimals by

$$(\log 2)/2^n + (\log 3)/3^n.$$

Using (2.8), we got the approximation

$$E = -1.33258\,22757\,33220\,87$$

which we checked by (5.2).

Incidentally, our values of $-\zeta'(n)/\zeta(n)$ check the seven-decimal values given in Walther [16].

With values for B, E, and e^{-C} now established, other computations could be undertaken. The second major computation cited in Section 1 was the tabulation by Rosser and Walker of many functions of primes for $x \leqq 16{,}000$. This verified Theorems 1–10 and their corollaries in this range, with the exception of Theorem 3, which was established for $p_n \leqq 16{,}000$. This computation also verified Theorems 18–23 for $x \leqq 16{,}000$.

The third major computation cited in Section 1 was the tabulation by Appel and Rosser of many functions of primes for $x \leqq 10^8$. This established Theorems 16 and 18–23 for $313 \leqq x \leqq 10^8$, which completed their verification. A discussion of this computation, together with a partial tabulation and many more details can be found in a report written by Appel and Rosser [19].

Theorem 17 is an immediate consequence of Theorem 16 as a result of Lemmas 5 and 6 of Rosser [12]. Thus Theorems 16 through 23 are established.

In Table II, at the end of this paper, we have listed values of $\vartheta(x)$ and other functions selected from the Rosser-Walker tabulation. If an isolated value is desired in the range $x \leqq 16{,}000$, it can be readily computed by working from the nearest entry in Table II; it can be checked by working from the entry on the other side. If numerous random values are desired, it is probably easier to generate an entire table on one of the modern very fast computers, using the entries in Table II to check key values. Other values for comparison are available in the report by Appel and Rosser [19].

For more limited ranges, one can derive values of some functions from tables already in the literature. Thus in Glaisher [4] is given a seven-decimal table of

$$f(x) = \textstyle\prod_{p \leqq x} (1 - 1/p)$$

and its common logarithm for $2 \leqq x \leqq 10{,}000$. By comparison with the Rosser-Walker tabulation, we verified that the Glaisher table is quite reliable. Round-off errors are common in the function values, but we found only four cases where the listed function value is in error by more than one unit in the last place. The correct values there are as follows:

$$f(4271) = 0.0670040, \qquad f(9397) = 0.0613076,$$

$$f(8609) = 0.0619246, \qquad f(9883) = 0.0609460.$$

In all these cases, the logarithmic values given were accurate to within round-off errors. The last column of our Table II gives selected values of $1/f(x)$ to ten decimals.

In Gram [5] is given an eight-decimal table of $\psi(x)$ for $x \leqq 2000$. The eighth decimal is quite unreliable, but no entry is in error by as much as 10^{-7}. From this, one can readily compute values of $\vartheta(x)$ for $x \leqq 2000$ by using our Table III, which gives the values of $\psi(x) - \vartheta(x)$ for $x < 50{,}653$. The arguments x are those prime powers p^r having $r \geqq 2$, so that the tabulated function is constant between entries. By means of Table III one can also compute $\psi(x)$ from the values of $\vartheta(x)$ given in Table II up to $x = 16{,}000$.

We now turn to Theorems 12–14, 24. We first take note of the following result from pp. 90–91 of Landau [7], vol. 1, proved by adapting the elementary derivation of Tchebichef [14].

THEOREM 25. *For* $1 \leqq x$

$$\vartheta(x) \leqq \psi(x) < 1.2\, ax + (3 \log x + 5)(\log x + 1), \tag{5.3}$$

where a denotes the constant

$$\tfrac{7}{15} \log 2 + \tfrac{3}{10} \log 3 + \tfrac{1}{6} \log 5 = 0.92129 \cdots,$$

according to the definition on p. 88 *of Landau* [7], *vol.* 1.

Using (5.3) with Theorem 18 gives the following weakened form of Theorem 9:

$$\vartheta(x) < 1.11\, x \qquad \text{for } 0 < x. \tag{5.4}$$

Using values of $\vartheta(x^{1/2})$ from the Rosser-Walker computation, we verified (3.38) for $x < 50{,}653$ by using Table III. For $50{,}653 \leqq x < 10^{24}$, one can verify (3.38) by (2.24) and the right side of (4.5); to do this, we proceed by cases such as $2^M \leqq x < 2^N$ where M and N are conveniently chosen integers. Finally, for $10^{24} \leqq x$, (3.38) holds by (2.24) and (5.4). Thus (3.38) has been completely established. From it, by (5.4) one can infer the following weakened form of (3.39):

$$\psi(x) - \vartheta(x) < 1.11\, x^{1/2} + 3x^{1/3} \qquad \text{for } 0 < x. \tag{5.5}$$

As (3.38) and the right side of (4.5) imply (4.12), we can complete the proof of Theorem 24 by establishing (4.11). This is readily done for $121 \leqq x < 50{,}653$ by means of Table III. By using Theorems 18 and 19 with (2.24), we can finish the proof of (4.11) and hence of Theorem 24.

By comparison with Gram's Table [5] of $\psi(x)$, Theorem 12 was verified for $x \leqq 2000$, and it was ascertained that in this range the maximum value of $\psi(x)/x$ lies between 1.03882 and 1.03883. Thus one can complete the proof of Theorem 12 by proving that $\psi(x) \leqq 1.03882\, x$ for $2000 \leqq x$. This follows for $x < 50{,}653$ by Theorem 18 and Table III. For $50{,}653 \leqq x \leqq 10^8$ it follows by Theorem 18 and (4.12). It will follow for all greater x by Table

I as soon as we have justified the values in this table, which we will do in the next section.

We can verify Theorem 13 in the range $0 < x < 50{,}653$ by reference to Table III. It then suffices to verify $\psi(x) - \vartheta(x) \leqq 1.42619\, x^{1/2}$ for $50{,}653 \leqq x$. For $10^6 \leqq x$, we infer this by (5.5). For $x < 10^6$ we use (2.25); the facts that $\psi(y) = 0$ for $y < 2$ and that ψ is monotone let us conclude

$$\begin{aligned}\psi(x) - \vartheta(x) &= \psi(x^{1/2}) + \psi(x^{1/3}) + \psi(x^{1/5}) - \psi(x^{1/6}) \\ &\qquad + \psi(x^{1/7}) - \psi(x^{1/10}) + \psi(x^{1/11}) + \psi(x^{1/13}) \\ &\qquad\qquad - \psi(x^{1/14}) - \psi(x^{1/15}) + \psi(x^{1/17}) + \psi(x^{1/19}) \\ &\leqq \psi(x^{1/2}) + \psi(x^{1/3}) + \psi(x^{1/5}) + \psi(x^{1/13}) \\ &< 1.04\, (x^{1/2} + x^{1/3} + x^{1/5} + x^{1/13})\end{aligned}$$

since (3.35) holds for $x \leqq 10^8$. This suffices to complete the proof.

At this point, therefore, we have established (2.10) through (2.14), Theorems 16 through 24, Theorem 12 except for $10^8 < x$, Theorem 13, and (3.38) of Theorem 14. We have also verified Theorems 1 through 10 and their corollaries for $x \leqq 16{,}000$ or for $p_n \leqq 16{,}000$.

6. Sharpening of some results of Rosser, with application to several proofs

In the preceding section, we carried our proofs as far as is practicable without appealing to very deep results. From here on, we shall be mainly concerned with invoking certain deep results to validate Table I and complete the proofs of the results stated in Section 3. Space does not permit us to give proofs in full, so that we shall assume that the reader is quite familiar with Ingham [6] and Rosser [12], from which we shall use notation and results with a minimum of reference.

The most significant sharpening of results from Rosser [12] arises from the fact that it is now known that the first 25000 zeros of the zeta function have real part equal to $\frac{1}{2}$, as shown in Lehmer [8] and Lehmer [9]. This enables us to replace the A on p. 223 of Rosser [12] by $A = e^{9.99}$. We do not now have $N(A) = F(A)$, which will make a slight change in a key formula, as we note below.

Observing that

$$322 + 546 \cos \phi + 329 \cos 2\phi + 130 \cos 3\phi + 25 \cos 4\phi = 2(1 + \cos \phi)^2\, (3 + 10 \cos \phi)^2 \geqq 0,$$

we can modify the proof of Theorem 20 of Rosser [12] to get a proof of

THEOREM 26. *For $A \leqq \gamma$, we have $\beta < 1 - 1/(R \log \gamma)$.*

The R here is that defined in Theorem 11. In other places, as here, it replaces the number 17.72 appearing in Rosser [12]. Thus, we are conforming with the notation of Rosser [12] when we temporarily abrogate the usual denotation of $\phi(n)$ and define

$$\phi(\gamma) = \phi(m, x, \gamma) = \gamma^{-m-1}\, e^{-(\log x)/(R \log \gamma)}. \tag{6.1}$$

However, for the purposes of the next theorem we consider (6.1) as defining $\phi(\gamma)$ for arbitrary positive R.

THEOREM 27. *If $\phi(\gamma)$ is defined as in* (6.1) *with m and R positive numbers, if* $2 \leqq K$, *and if* $0 \leqq \log x \leqq (m+1)\, R \log^2 K$, *then*

$$\sum_{K<\gamma} \phi(\gamma) < 2R(K)\phi(K) + Q \int_K^\infty \phi(y) \log \frac{y}{2\pi}\, dy, \tag{6.2}$$

where

$$Q = \frac{1}{2\pi} + \frac{0.137 \log K + 0.443}{K \log K \log (K/2\pi)}. \tag{6.3}$$

Proof. Proceed as in the proof of Lemma 18 of Rosser [12].

COROLLARY. *If in* (6.1) *we take R to be the R of Theorem* 11, *and if* $A \leqq K$, $0 < m$, *and*

$$0 \leqq \log x < 1748(m+1), \tag{6.4}$$

then

$$\sum_{K<\gamma} \phi(\gamma) < 2R(K)\phi(K) + 0.1592 \int_K^\infty \phi(y) \log \frac{y}{2\pi}\, dy. \tag{6.5}$$

Proof. As we are here using the R of Theorem 11, (6.4) verifies the final hypothesis of Theorem 27. In (6.5), the coefficient in front of the integral is got by taking $K = e^{9.99}$ in (6.3), which is permissible since $e^{9.99} = A \leqq K$.

In Rosser [12], in the situation corresponding to taking $K = A$ in the corollary, the coefficient 2 did not appear in the first term on the right of (6.5). This is because $N(A) = F(A)$ in that paper. Except for that, we now proceed as in the proofs of Lemma 19 and Theorem 21 of Rosser [12] to derive

THEOREM 28. *If m is a positive integer,*

$$\sum_\rho \frac{1}{|\gamma^{m+1}|} \leqq k_m, \qquad \log a < \frac{1748 m^2}{m + 0.123},$$

$$\delta \geqq 2 \left\{ \frac{k_m}{a^{1/2}} + \frac{0.0003647\, m^2 + 1.298\, m + 0.1592}{\left(1 - \dfrac{m + 0.123}{1748 m^2} \log a\right) m^2 A^m a^{1/175}} \right\}^{1/(m+1)},$$

$$\varepsilon = \frac{\delta}{2} \left\{ \left(\frac{(1+\delta)^{m+1} + 1}{2} \right)^m + m \right\},$$

and $1 + m\delta a < a$, *then for* $a \leqq x$

$$x(1-\varepsilon) - 1.84 < \psi(x) < x(1+\varepsilon) - \tfrac{1}{2} \log (1 - x^{-2}).$$

By taking a successively equal to e^b for the various values of b listed in Table I, and using with these values the listed values of m, the values of ε listed in Table I were derived from Theorem 28. In particular, the value for $b = 5000$ was listed just as it was given by Theorem 28 despite the fact that Theorem 28 gives a smaller ε for $b = 4900$, as listed in Table I. The bounds used for k_1, k_2, and k_3 are those given in Lemma 17 of Rosser [12]. Bounds for larger m were obtained by the trivial inequality

$$14 \sum |\gamma^{-m-2}| < \sum |\gamma^{-m-1}|.$$

Now that we have justified Table I, we use it to complete the proof of Theorem 12, as noted above.

Turning to Theorem 9, we verify it for $x \leqq 10^8$ by Theorem 18. For $10^8 \leqq x \leqq 10^{16}$, we have $\vartheta(x) < \psi(x) - x^{1/2}$ by Theorem 24, and so verify Theorem 9 in this range by Table I. Above 10^{16}, we use Table I with the trivial inequality $\vartheta(x) < \psi(x)$.

We complete the verification of Theorem 10 for $16{,}000 \leqq x \leqq 10^8$ by Theorem 19. Above this, we use (5.5) with Table I.

As far as it furnishes bounds on $\vartheta(x)$, we verify Theorem 11 for $x \leqq 101$ by comparison with values of $\vartheta(x)$ taken from the Rosser-Walker tabulation. Now with ε defined as in Theorem 11, we have $\varepsilon \geqq 0.625$ for $2 \leqq x \leqq e^{9R}$. Thus we can complete the verification of Theorem 11 in this range by Theorems 10 and 12. This puts us in the range of Table I. From here to e^{4800} we can proceed by using Table I with (5.5). We now complete the proof of Theorem 11 as in the proof of Theorem 22 of Rosser [12], noting that by (5.5) the difference between $\vartheta(x)$ and $\psi(x)$ is so small as to be more than allowed for by the fact that various quantities do not actually attain the upper bounds by which they are replaced in the proof.

We verify (3.37) for $x \leqq 10^{16}$ by Theorem 24. For greater x, it follows by (2.24) and Theorem 10. Thus we have completed the proof of Theorem 14.

For some of our later proofs we will need results that are sharper in certain ranges than Theorem 4. We now state and prove several such results.

THEOREM 29. *For* $1451 \leqq x \leqq e^{375}$,

$$x\left(1 - \frac{0.31}{\log x}\right) < \vartheta(x) < x\left(1 + \frac{0.31}{\log x}\right). \tag{6.6}$$

THEOREM 30. *For* $809 \leqq x \leqq e^{575}$,

$$x\left(1 - \frac{0.40}{\log x}\right) < \vartheta(x) < x\left(1 + \frac{0.40}{\log x}\right). \tag{6.7}$$

THEOREM 31. *For* $569 \leqq x$,

$$x\left(1 - \frac{0.47}{\log x}\right) < \vartheta(x) < x\left(1 + \frac{0.47}{\log x}\right). \tag{6.8}$$

For $x \leqq 16{,}000$, these are established by means of the Rosser-Walker tabulation. For 16,000 $\leqq x \leqq 10^8$, these are established by Theorems 18 and 19. For $10^8 \leqq x \leqq 10^{16}$, we use Theorem 24 and Table I. For $10^{16} \leqq x \leqq e^{5000}$, we use the corollary of Theorem 14 and Table I. Finally, above e^{5000} we use Theorem 11.

From these, Theorem 4 is an easy consequence.

7. Proof of Theorems 1 through 3

We start with five lemmas. As their proofs are similar, we state all five lemmas first before giving the proofs. We first make the definition

$$J(x, a) = \pi(1451) - \frac{\vartheta(1451)}{\log 1451} + \frac{x}{\log x}\left(1 + \frac{a}{\log x}\right) + \int_{1451}^{x}\left(1 + \frac{a}{\log y}\right)\frac{dy}{\log^2 y}. \tag{7.1}$$

LEMMA 1. *For $e^8 \leqq x$,*

$$\mathrm{li}(x) < \frac{x}{\log x}\left(1 + \frac{3}{2 \log x}\right). \tag{7.2}$$

LEMMA 2. *For $10^8 \leqq x$ and $a = 0.31$,*

$$J(x, a) < \frac{x}{\log x}\left(1 + \frac{3}{2 \log x}\right). \tag{7.3}$$

LEMMA 3. *For $e^{100} \leqq x$ and $a = 0.47$, the inequality (7.3) is valid.*

LEMMA 4. *For $e^5 \leqq x$,*

$$x/(\log x - \tfrac{1}{2}) < \mathrm{li}(x) - \mathrm{li}(x^{1/2}). \tag{7.4}$$

LEMMA 5. *For $10^8 \leqq x$ and $a = -0.47$,*

$$x/(\log x - \tfrac{1}{2}) < J(x, a). \tag{7.5}$$

For each of these lemmas, the proof is in two parts. First, one verifies that in the stated range of x, the derivative of the left side is less than that of the right side. Second, one verifies that at the lower limit of x the left side is less than the right side. To perform the needed calculations, one can use the reduction formula

$$\int_b^x \frac{a\, dy}{\log^{a+1} y} = \frac{b}{\log^a b} - \frac{x}{\log^a x} + \int_b^x \frac{dy}{\log^a y} \tag{7.6}$$

to express the various integrals in terms of $\mathrm{li}(x)$ and elementary functions. For $x \leqq e^{15}$, one can get numerical values of $\mathrm{li}(x)$ from the tables of $\mathrm{Ei}(y)$ given in [18]. Outside this range, one can appeal to the following result.

THEOREM 32. *If m is a positive integer and $m \leqq y \leqq m + 1$, then*

$$\frac{2}{3}\left(\frac{2\pi}{m}\right)^{1/2} - \frac{1.06}{m} + e^y \sum_{j=1}^{m-1} \frac{(j-1)!}{y^j} \tag{7.7}$$

and

$$\frac{2}{3}\left(\frac{2\pi}{m+1}\right)^{1/2} + \frac{1.06}{m+1} + e^y \sum_{j=1}^{m} \frac{(j-1)!}{y^j} \tag{7.8}$$

are lower and upper bounds respectively for $\mathrm{Ei}(y)$.

This is a consequence of equations (58), (68), and (69), and of Lemma 3 and Theorem 7 of Rosser [13].

We note the value $\pi(1451) = 230$ and the approximation $\vartheta(1451) = 1396.4$ taken from the Rosser-Walker tabulation.

We now turn to (3.2), which has already been established for $x \leqq 16{,}000$ by the Rosser-Walker tabulation. By Lemma 1 and Theorem 16, we verify (3.2) up to $x = 10^8$. We have by (4.17) that

$$\pi(x) = \pi(1451) - \frac{\vartheta(1451)}{\log 1451} + \frac{\vartheta(x)}{\log x} + \int_{1451}^{x} \frac{\vartheta(y)\,dy}{y \log^2 y}.$$

By Theorem 29, we can conclude $\pi(x) < J(x, 0.31)$ if $1451 \leqq x \leqq e^{375}$. So by Lemma 2 we can infer (3.2) for $10^8 \leqq x \leqq e^{375}$. In a similar way we combine Lemma 3 and Theorem 31 to complete the proof of (3.2).

In a similar way, we combine Lemma 4 and Theorem 16 to verify (3.3) for $x \leqq 10^8$, and complete the verification by combining Lemma 5 and Theorem 31.

We get (3.1) from (3.3), and (3.4) from (3.2), by applying the inequality

$$\frac{x}{\log x}\left(1 + \frac{a}{\log x}\right) < \frac{x}{\log x - a} \qquad \text{for } e^a < x$$

in the two cases $a = \frac{1}{2}$ and $a = \frac{3}{2}$.

As preliminaries for the proof of Theorem 3 we undertake the proof of (3.12) and

$$p_n < n(\log n + 2 \log\log n) \qquad \text{for} \quad 4 \leqq n. \tag{7.9}$$

These were proved in Rosser [11], but can be derived so readily from the strong results now available that it seems worthwhile to indicate new proofs by this method. For instance, suppose if possible that $p_n \leqq n \log n$. Then

$$n \leqq \pi(n \log n).$$

So by (3.2), we have

$$n < \frac{n \log n}{\log n + \log\log n}\left(1 + \frac{1.5}{\log n + \log\log n}\right),$$

a result which certainly fails if $e^5 \leqq n$. So (3.12) holds for $e^5 \leqq n$, and a trivial computation verifies it for smaller n. The proof of (7.9) is analogous.

From these, we can now infer Lemmas 9 and 10 of Rosser [12], using the proofs given there. Using Lemma 9 of Rosser [12] with Theorem 18 gives

$$n \log n + n \log\log n - n - \mathrm{li}(n) < p_n$$

for $5 \leqq n \leqq \pi(10^8)$. Then (3.10) follows for $e^4 \leqq n \leqq \pi(10^8)$ by Lemma 7 of Rosser [12]. Now use Lemma 9 of Rosser [12] with Theorem 30, and infer

$$n \log n + n \log\log n - n - \mathrm{li}(n) < p_n\left(1 + \frac{0.40}{\log p_n}\right)$$

for $140 \leqq n \leqq e^{568}$. As $\mathrm{li}(n) < 0.1\, n$ for $\pi(10^8) \leqq n$, by Lemma 7 of Rosser [12], and

$$p_n/(\log p_n) < n \tag{7.10}$$

by (3.5), we infer (3.10) for $n \leqq e^{568}$. We use Theorem 31 in a similar manner to complete the proof of (3.10).

We next prove (3.11). We note first of all that it has been verified for $n \leqq 1862$ by the Rosser-Walker tabulation. Now let $1862 \leqq n$, and suppose that (3.11) has been verified for all integers less than n. Then the hypothesis of Lemma 10 of Rosser [12] is verified, and we conclude that

$$\vartheta(p_n) < n \log n + n \log\log n - n + \frac{n \log\log n}{\log n}. \tag{7.11}$$

By Theorem 19, if $n \leqq \pi(10^8)$, then

$$p_n - 2(p_n)^{1/2} < n \log n + n \log\log n - n + \frac{n \log\log n}{\log n}.$$

By (7.9),

$$2(p_n)^{1/2} < 2(n \log n + 2n \log\log n)^{1/2} < 0.2\, n,$$

so that (3.11) is verified. Now let $\pi(10^8) \leqq n \leqq e^{369}$. Then (7.11) and Theorem 29 give

$$p_n\left(1 - \frac{0.31}{\log p_n}\right) < n \log n + n \log\log n - n + \frac{n \log\log n}{\log n}.$$

Using (7.10), we again infer (3.11). In a similar way, we can use Theorem 31 to verify (3.11) for $e^{369} \leqq n$.

8. Proof of Theorems 5 through 8

We require several lemmas.

LEMMA 6. *For* $1 \leqq x$ *and* $A \leqq K$,

$$K(1, x) < \frac{0.0463}{x^{1/2}} + \frac{2R(K)}{K^2} + 0.1592\, \frac{1 + \log(K/2\pi)}{K} + e^{-9.61}\, x^{-1/(R \log K)}.$$

Proof. Taking $m = 1 = x$ in (6.5) gives

$$\sum_{K<\gamma} \frac{1}{\gamma^2} < \frac{2R(K)}{K^2} + 0.1592\, \frac{1 + \log(K/2\pi)}{K}. \tag{8.1}$$

Taking $K = A$ in this gives

$$\sum_{A<\gamma} \frac{1}{\gamma^2} < e^{-9.61}. \tag{8.2}$$

By Lemmas 16 and 17 of Rosser [12]

$$\begin{aligned} K(1,x) &< \frac{0.0463}{x^{1/2}} + \sum_{A<\gamma\leqq K} \phi(\gamma) + \sum_{K<\gamma} \phi(\gamma) \\ &< \frac{0.0463}{x^{1/2}} + \sum_{A<\gamma\leqq K} \frac{x^{-1/(R\log K)}}{\gamma^2} + \sum_{K<\gamma} \frac{1}{\gamma^2}. \end{aligned}$$

Our lemma now follows by use of (8.1) and (8.2).

We next state two lemmas whose proofs are so similar that we give only the more difficult one, namely that for Lemma 8.

LEMMA 7. *For* $1 < x$,

$$\left| \int_x^\infty \frac{y-\psi(y)}{y^2}\, dy \right| < K(1,x) + \frac{1.84}{x} + \frac{0.31}{x^3}.$$

LEMMA 8. *For* $1 < x$,

$$\left| \int_x^\infty \frac{(y-\psi(y))\,(1+\log y)}{y^2 \log^2 y}\, dy \right| < \frac{2+\log x}{\log^2 x}\left(K(1,x) + \frac{1.84}{x} + \frac{0.31}{x^3}\right).$$

Proof. We have

$$\int_x^\infty \frac{y^{\rho-2}(1+\log y)}{\log^2 y}\, dy = \frac{-x^{\rho-1}}{(\rho-1)\log x} - \frac{\rho}{(\rho-1)^2}\left\{\frac{x^{\rho-1}}{\log^2 x} - 2\int_x^\infty \frac{y^{\rho-2}}{\log^3 y}\, dy\right\}.$$

Now

$$\begin{aligned} \left| \frac{x^{\rho-1}}{\log^2 x} - 2\int_x^\infty \frac{y^{\rho-2}}{\log^3 y}\, dy \right| &\leqq \frac{x^{\beta-1}}{\log^2 x} + 2\int_x^\infty \frac{y^{\beta-2}}{\log^3 y}\, dy \\ &< \frac{x^{\beta-1}}{\log^2 x} + 2x^{\beta-1}\int_x^\infty \frac{dy}{y\log^3 y} = \frac{2x^{\beta-1}}{\log^2 x}. \end{aligned}$$

So

$$\begin{aligned} \left| \frac{1}{\rho}\int_x^\infty \frac{y^{\rho-2}(1+\log y)}{\log^2 y}\, dy \right| &< \frac{1}{|\rho(\rho-1)|}\frac{x^{\beta-1}}{\log x} + \frac{1}{|\rho-1|^2}\frac{2x^{\beta-1}}{\log^2 x} \\ &< \frac{2+\log x}{\log^2 x}\cdot\frac{x^{\beta-1}}{\gamma^2}. \end{aligned}$$

Hence

$$\sum_\rho \left| \frac{1}{\rho}\int_x^\infty \frac{y^{\rho-2}(1+\log y)}{\log^2 y}\, dy \right| < \frac{2+\log x}{\log^2 x} K(1,x). \tag{8.3}$$

So by Theorem 29 on p. 77 of Ingham [6], we have

$$\left| \int_x^\infty \frac{(y-\psi(y))\,(1+\log y)}{y^2\log^2 y}\, dy \right| < \frac{2+\log x}{\log^2 x} K(1,x) + I,$$

where

$$\begin{aligned}
I &= \int_x^\infty \frac{(2\log 2\pi - \log(1 - y^{-2}))(1 + \log y)}{2y^2 \log^2 y}\, dy \\
&< \frac{1 + \log x}{2\log^2 x} \int_x^\infty \frac{2\log 2\pi - \log(1 - y^{-2})}{y^2}\, dy \\
&= \frac{1 + \log x}{2\log^2 x} \left\{ \frac{2\log 2\pi}{x} + \sum_{r=1}^{\infty} \frac{x^{-2r-1}}{r(2r+1)} \right\} \\
&< \frac{1 + \log x}{\log^2 x} \left\{ \frac{\log 2\pi}{x} + \sum_{r=1}^{\infty} \frac{x^{-3}}{2r(2r+1)} \right\} \\
&< \frac{2 + \log x}{\log^2 x} \left\{ \frac{\log 2\pi}{x} + \frac{1 - \log 2}{x^3} \right\}.
\end{aligned}$$

From this, the lemma follows.

LEMMA 9. *For* $0 \leqq a < n$,

$$\int_x^\infty y^{a-n-1} \frac{1 + n\log y}{\log^2 y}\, dy \leqq \frac{n}{n-a} \cdot \frac{x^{a-n}}{\log x}.$$

Proof. We have

$$\begin{aligned}
\int_x^\infty y^{a-n-1} \frac{1 + n\log y}{\log^2 y}\, dy &= -\int_x^\infty y^a \frac{d}{dy}\left(\frac{y^{-n}}{\log y}\right) dy \\
&= \frac{x^{a-n}}{\log x} + a \int_x^\infty \frac{y^{a-n-1}}{\log y}\, dy \\
&\leqq \frac{x^{a-n}}{\log x} + \frac{a}{\log x} \int_x^\infty y^{a-n-1}\, dy \\
&= \frac{n}{n-a} \cdot \frac{x^{a-n}}{\log x}.
\end{aligned}$$

Let us define

$$\begin{aligned}
L(x) &= \frac{2 + \log x}{\log x}\left(K(1, x) + \frac{1.84}{x} + \frac{0.31}{x^3}\right) \\
&\qquad + \frac{2.04}{x^{1/2}} + 4.5\, x^{-2/3} + \frac{1.02}{x - 1},
\end{aligned} \tag{8.4}$$

$$M(x, a) = (\log x) \log\left(1 + \frac{1}{2\log^2 x}\right) - \frac{a}{\log x}. \tag{8.5}$$

LEMMA 10. *If* $1 < A \leqq B$ *and* $a < \frac{1}{2}$ *and*

$$\log A \geqq \left(\frac{1}{2} \cdot \frac{1 + 2a}{1 - 2a}\right)^{1/2} \tag{8.6}$$

and $L(A) < M(B, a)$, *then* $L(x) < M(x, a)$ *for* $A \leqq x \leqq B$.

Proof. We readily verify that for $A \leqq x$ both $L(x)$ and $M(x, a)$ are decreasing functions of x. So for $A \leqq x \leqq B$

$$L(x) \leqq L(A) < M(B, a) \leqq M(x, a).$$

LEMMA 11. *For* $10^8 \leqq x \leqq e^{600}$,

$$L(x) < M(x, 0.31). \tag{8.7}$$

Proof. By (8.4) and (8.5) and by taking $K = \infty$ in Lemma 6, we find that

$$L(10^8) < 3 \times 10^{-4} < M(e^{600}, 0.31).$$

So we can use Lemma 10 with $A = 10^8$, $B = e^{600}$.

LEMMA 12. *For* $e^{375} \leqq x$,

$$L(x) < M(x, 0.47). \tag{8.8}$$

Proof. By taking $K = e^{25}$ in Lemma 6, we find that

$$K(1, x) < 0.0463/x^{1/2} + e^{-9.61}\, x^{-1/(25R)} + 10^{-10}.$$

From this, we readily verify by (8.4) and (8.5) that

$$\begin{aligned} L(e^{375}) &< 2.9 \times 10^{-5} < M(e^{1000}, 0.47), \\ L(e^{1000}) &< \quad 7 \times 10^{-6} < M(e^{4000}, 0.47), \\ L(e^{4000}) &< \quad 9 \times 10^{-9} < M(e^{3000000}, 0.47). \end{aligned}$$

Then by three applications of Lemma 10, we verify (8.8) for $e^{375} \leqq x \leqq e^{3000000}$. Finally for $e^{3000000} \leqq x$, we take

$$K = \exp \frac{\log x}{R(\log\log x - \log 400)}$$

and observe that

$$L(x) < 0.028/\log x < M(x, 0.47)$$

by (8.4), Lemma 6, and (8.5).

LEMMA 13. *For* $10^8 \leqq x$,

$$\left| \sum_{p \leqq x} \frac{1}{p} - \log\log x - B \right| < \log\left(1 + \frac{1}{2\log^2 x}\right) - \frac{1.02}{(x-1)\log x}. \tag{8.9}$$

Proof. By (4.20), Lemma 8, and (3.39), we have

$$\begin{aligned} &\left| \sum_{p \leqq x} \frac{1}{p} - \log\log x - B \right| \\ &\leqq \left| \frac{\vartheta(x) - x}{x \log x} \right| + \left| \int_x^\infty \frac{(y - \psi(y))\,(1 + \log y)}{y^2 \log^2 y}\, dy \right| \\ &\qquad + \left| \int_x^\infty \frac{(\psi(y) - \vartheta(y))\,(1 + \log y)}{y^2 \log^2 y}\, dy \right| \\ &< \left| \frac{\vartheta(x) - x}{x \log x} \right| + \frac{2 + \log x}{\log^2 x}\left(K(1, x) + \frac{1.84}{x} + \frac{0.31}{x^3}\right) \\ &\qquad + \int_x^\infty \frac{(1.02\, y^{1/2} + 3y^{1/3})(1 + \log y)}{y^2 \log^2 y}\, dy. \end{aligned}$$

We apply Lemma 9 with $n = 1$, once with $a = \frac{1}{2}$ and once with $a = \frac{1}{3}$. Then, with the help of (8.4), the above inequality reduces to

$$\left| \sum_{p \leqq x} \frac{1}{p} - \log \log x - B \right| < \left| \frac{\vartheta(x) - x}{x \log x} \right| + \frac{L(x)}{\log x} - \frac{1.02}{(x-1) \log x}.$$

From this the lemma follows; we first assume $10^8 \leqq x \leqq e^{375}$ and use Theorem 29, Lemma 11, and (8.5), and we then assume $e^{375} \leqq x$ and use Theorem 31, Lemma 12, and (8.5).

We now proceed to the proof of Theorem 5. It was proved for $x \leqq 16{,}000$ by the Rosser-Walker tabulation. For $16{,}000 \leqq x \leqq 10^8$, it follows from Theorem 20. Finally, for $10^8 \leqq x$, it follows by Lemma 13.

The proof of Theorem 6 proceeds similarly. For $10^8 \leqq x$, it depends on a lemma analogous to Lemma 13 that uses Lemma 7 rather than Lemma 8. The entire proof parallels that of Theorem 5 so closely that we omit the details.

For $x \leqq 16{,}000$, Theorem 7 and Theorem 8 follow from the Rosser-Walker tabulation, and for $16{,}000 \leqq x \leqq 10^8$, they follow by Theorem 23. So let us assume that $10^8 \leqq x$. We apply Lemma 13 in the form that the left side of (8.9) with the absolute value bars removed is greater than the negative of the right side. In this, we substitute for B from (2.7) and take the exponential of both sides. We get:

$$(8.10) \quad \frac{e^{-C}}{\log x}\left(1 + \frac{1}{2 \log^2 x}\right) > \left\{\prod_{p \leqq x}\left(1 - \frac{1}{p}\right)\right\} \exp\left(\frac{1.02}{(x-1)\log x} + S\right),$$

where

$$S = \sum_{x<p}\left\{\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right\} = -\sum_{n=2}^{\infty} \frac{1}{n} \sum_{x<p} \frac{1}{p^n}.$$

Taking $f(x) = x^{-n}$ in (4.13), we infer if $n > 1$

$$\sum_{x<p} \frac{1}{p^n} = -\frac{\vartheta(x)}{x^n \log x} + \int_x^\infty \frac{\vartheta(y)}{y^{n+1}} \cdot \frac{1 + n \log y}{\log^2 y}\, dy.$$

Then by Theorem 9 and Lemma 9

$$\sum_{x<p} \frac{1}{p^n} < 1.02 \int_x^\infty y^{-n} \frac{1 + n \log y}{\log^2 y}\, dy$$

$$\leqq \frac{1.02n}{n-1} \cdot \frac{x^{1-n}}{\log x} \leqq \frac{1.02n}{x^{n-1} \log x}.$$

So

$$S > -\sum_{n=2}^{\infty} \frac{1.02}{x^{n-1} \log x} = \frac{-1.02}{(x-1) \log x}.$$

So by (8.10) we conclude that (3.26) holds. Inverting both sides gives (3.28).

Similarly, if we remove the absolute value bars on the left side of (8.9) and take the exponential of both sides, we infer (3.29). Inverting both sides gives (3.25).

9. Proof of Theorem 15

In this section, $\phi(n)$ again denotes the Euler totient function.

LEMMA 14. *If m and n are positive integers, and $n < \exp\vartheta(p_{m+1})$, then*

$$n/\phi(n) \leqq \prod_{p\leqq p_m} p/(p-1). \tag{9.1}$$

Proof. Let $q_1, q_2, \cdots, q_r$ be the distinct primes, in increasing order, which divide n. Then

$$\exp\vartheta(p_{m+1}) > n \geqq q_1 \cdots q_r \geqq p_1 \cdots p_r = \exp\vartheta(p_r).$$

So $r \leqq m$. Consequently

$$n/\phi(n) = \prod_{s=1}^{r} q_s/(q_s - 1) \leqq \prod_{s=1}^{r} p_s/(p_s - 1) \leqq \prod_{s=1}^{m} p_s/(p_s - 1),$$

which is the same as (9.1)

By means of Lemma 14 we can verify (3.41) numerically for a succession of intervals if $\log n$ is not great. For instance, we readily verify numerically that

$$\prod_{p\leqq 7} p/(p-1) = 4.375 < e^C \log\log 210 + 5/(2\log\log 210).$$

So we can take $m = 4$ in Lemma 14, and conclude that if $210 \leqq n < 2310 = \exp\vartheta(p_{m+1})$, then

$$n/\phi(n) < e^C \log\log 210 + 5/(2\log\log 210) \leqq e^C \log\log n + 5/(2\log\log n)$$

With the help of values of $\vartheta(x)$ and of

$$\prod_{p\leqq x} p/(p-1)$$

taken from the Rosser-Walker tabulation, we proceeded in a step-by-step manner as indicated above, verifying (3.41) for $3 \leqq n < \exp\vartheta(313)$ except at

$$n = \prod_{p\leqq 23} p,$$

at which point (3.42) holds. As $294 < \vartheta(313)$, this verifies Theorem 15 for $n \leqq e^{294}$.

THEOREM 33. *If $x \geqq 5$ and $n < \exp\vartheta(x)$, then*

$$n/\phi(n) \leqq \prod_{p\leqq x-2} p/(p-1). \tag{9.2}$$

Proof. Choose m so that $p_{m+1} \leqq x < p_{m+2}$. Then $n < \exp\vartheta(x) = \exp\vartheta(p_{m+1})$. Also $5 \leqq p_{m+1}$, else we would have $p_{m+2} \leqq 5$, contradicting $5 \leqq x < p_{m+2}$. So $p_m \leqq p_{m+1} - 2 \leqq x - 2$. Now use Lemma 14.

LEMMA 15. *Let n be an integer greater than unity and y a real number such that $288 \leqq \log n + y$ and $\log n < \vartheta(\log n + y)$, and*

$$0 \leqq y - 2 \leqq (0.9\log n)/\log\log n.$$

Then (3.41) holds for this value of n.

Proof. By (3.29) and Theorem 33

$$\begin{aligned}\frac{e^{-C}n}{\phi(n)} &< \log(\log n + y - 2) + \frac{0.5}{\log(\log n + y - 2)} \\ &\leqq \log\log n + \log\left(1 + \frac{y-2}{\log n}\right) + \frac{0.5}{\log\log n} \\ &\leqq \log\log n + \frac{y-2}{\log n} + \frac{0.5}{\log\log n} \\ &\leqq \log\log n + \frac{1.4}{\log\log n}.\end{aligned}$$

From this, (3.41) follows.

To complete the proof of Theorem 15, we have to show that (3.41) holds for $294 \leqq \log n$. In fact from Lemma 15, one can deduce (3.41) for $255 \leqq \log n$. First assume $255 \leqq \log n \leqq 1340$. Take y to be

$$2 + 2(1 + \log n)^{1/2}.$$

Then we have $288 < \log n + y < 1420$. Also

$$\log n = (\log n + y) - 2(\log n + y)^{1/2},$$

so that $\log n < \vartheta(\log n + y)$ by Theorem 19. As we easily verify that $0 \leqq y - 2 \leqq (0.9 \log n)/\log\log n$, we infer (3.41) by Lemma 15. Finally, let $1340 \leqq \log n$. Take $y = (0.9 \log n)/\log\log n$. Then

$$\log n < \vartheta(\log n + y)$$

holds by (3.14), and the other hypotheses of Lemma 15 are readily verified, giving (3.41) again.

TABLE I

$(1 - \varepsilon)\, x < \psi(x) < (1 + \varepsilon)\, x$ for $e^b \leqq x$

b	m	$10^n\varepsilon$	n	b	m	$10^n\varepsilon$	n	b	m	$10^n\varepsilon$	n
18.4	1	1.6327	2	62	4	1.3108	3	725	5	5.9384	4
18.5	1	1.6295	2	64	4	1.2825	3	750	5	5.8017	4
18.6	2	1.6256	2	66	4	1.2696	3	775	5	5.6682	4
18.7	2	1.5987	2	68	4	1.2630	3	800	5	5.5378	4
18.8	2	1.5722	2	70	4	1.2588	3	825	5	5.4103	4
18.9	2	1.5462	2	72	4	1.2555	3	850	4	5.2843	4
19.0	2	1.5206	2	74	5	1.2138	3	875	4	5.1397	4
19.5	2	1.3993	2	76	5	1.1419	3	900	4	4.9991	4
20.0	2	1.2880	2	78	5	1.1074	3	925	4	4.8624	4
20.5	2	1.1861	2	80	5	1.0920	3	950	4	4.7294	4
21	2	1.0928	2	85	5	1.0793	3	975	4	4.6001	4
22	2	9.2933	3	90	5	1.0737	3	1000	4	4.4744	4
23	2	7.9327	3	95	6	1.0075	3	1050	4	4.2331	4
24	2	6.8090	3	100	6	9.9653	4	1100	4	4.0049	4
25	2	5.8927	3	105	6	9.9194	4	1150	3	3.7703	4
26	2	5.1592	3	110	6	9.8791	4	1200	3	3.5217	4
27	2	4.5870	3	125	6	9.7608	4	1300	3	3.0730	4
28	2	4.1548	3	150	6	9.5669	4	1400	3	2.6819	4
29	2	3.8400	3	175	6	9.3768	4	1500	3	2.3411	4
30	2	3.6192	3	200	6	9.1904	4	1600	2	2.0120	4
31	2	3.4690	3	225	6	9.0078	4	1800	2	1.4334	4
32	2	3.3691	3	250	6	8.8289	4	2000	2	1.0274	4
33	2	3.3034	3	275	6	8.6535	4	2200	2	7.4229	5
34	2	3.2601	3	300	6	8.4816	4	2400	2	5.4246	5
35	2	3.2310	3	325	6	8.3131	4	2600	2	4.0328	5
36	2	3.2110	3	350	6	8.1480	4	2800	2	3.0861	5
37	2	3.1964	3	375	6	7.9861	4	3000	2	2.5073	5
38	2	3.1853	3	400	6	7.8275	4	3200	3	2.4309	5
39	3	2.9365	3	425	6	7.6721	4	3400	3	1.8801	5
40	3	2.6399	3	450	6	7.5197	4	3600	3	1.4596	5
42	3	2.2000	3	475	6	7.3704	4	3800	3	1.1388	5
44	3	1.9387	3	500	6	7.2240	4	4000	3	8.9428	6
46	3	1.8064	3	525	6	7.0806	4	4200	3	7.0899	6
48	3	1.7467	3	550	6	6.9400	4	4400	3	5.7041	6
50	3	1.7202	3	575	6	6.8023	4	4500	3	5.1602	6
52	3	1.7072	3	600	6	6.6672	4	4600	3	4.7095	6
54	3	1.6994	3	625	5	6.5182	4	4700	3	4.3563	6
56	3	1.6936	3	650	5	6.3682	4	4800	3	4.1232	6
58	4	1.5013	3	675	5	6.2216	4	4900	3	4.0977	6
60	4	1.3740	3	700	5	6.0783	4	5000	3	4.9163	6

TABLE II

x	$\vartheta(x)$	$\sum_{p \leqq x} \frac{1}{p}$	$\sum_{p \leqq x} \frac{\log p}{p}$	$\prod_{p \leqq x} \frac{p}{p-1}$
500	474.55444 41547	2.09670 95528	4.94448 99600	11.15950 15857
1000	956.24526 51201	2.19808 01272	5.60951 04754	12.35097 56739
1500	1462.14165 18014	2.25562 82528	6.01869 61634	13.08291 09945
2000	1939.83920 03026	2.29244 84920	6.29327 07024	13.57375 00182
2500	2433.60275 29800	2.32105 31990	6.51384 37141	13.96771 93817
3000	2932.35921 18787	2.34404 93716	6.69584 35999	14.29270 53203
3500	3409.45718 45205	2.36222 13278	6.84275 32932	14.55484 67736
4000	3911.14539 95812	2.37858 30199	6.97729 51026	14.79498 00928
4500	4412.18831 05019	2.39276 53465	7.09571 03205	15.00632 74987
5000	4911.69535 17069	2.40518 86577	7.20087 63227	15.19393 85100
5500	5391.37223 83531	2.41586 31315	7.29230 15305	15.35701 01674
6000	5893.29745 72481	2.42598 40781	7.37988 11065	15.51324 05518
6500	6408.90736 71752	2.43543 75880	7.46249 21364	15.66060 23831
7000	6920.42102 99437	2.44401 57706	7.53814 09953	15.79552 97610
7500	7364.85741 60237	2.45091 39779	7.59945 43711	15.90487 48550
8000	7875.15038 47974	2.45829 17384	7.66550 12982	16.02265 87938
8500	8343.99966 34035	2.46460 59355	7.72243 05072	16.12415 52820
9000	8870.37499 26578	2.47124 44465	7.78267 62189	16.23155 79133
9500	9418.36877 33985	2.47772 62760	7.84187 45368	16.33711 55421
10000	9895.99137 91570	2.48305 99472	7.89086 36043	16.42448 96322
10500	10403.90704 75207	2.48842 73950	7.94043 00603	16.51288 85620
11000	10877.34163 04695	2.49317 02420	7.98445 72244	16.59139 63457
11500	11362.43971 33403	2.49778 96786	8.02755 03003	16.66821 99665
12000	11840.48575 38722	2.50212 24249	8.06816 24606	16.74059 88963
12500	12348.83694 44657	2.50652 81281	8.10963 67280	16.81451 87368
13000	12868.72809 74239	2.51084 52555	8.15044 40734	16.88726 89363
13500	13371.76845 32826	2.51484 72175	8.18842 74850	16.95498 91186
14000	13867.29252 76925	2.51862 72664	8.22444 83834	17.01920 34304
14500	14307.28400 32521	2.52185 38317	8.25531 02383	17.07420 76210
15000	14844.79169 21653	2.52565 30642	8.29177 62756	17.13920 20969
15500	15384.23856 36932	2.52932 29775	8.32712 81756	17.20221 91093
16000	15886.79246 84213	2.53262 50069	8.35904 03921	17.25911 70367

TABLE III

n	$\psi(n) - \vartheta(n)$	n	$\psi(n) - \vartheta(n)$
4	0.69314 71805 59945	5041	90.52702 04080 85442
8	1.38629 43611 19891	5329	94.81747 98492 33834
9	2.48490 66497 88000	6241	99.18692 77017 00855
16	3.17805 38303 47946	6561	100.28553 99903 68965
25	4.78749 17427 82046	6859	103.22997 89695 35405
27	5.88610 40314 50156	6889	107.64881 95773 32003
32	6.57925 12120 10101	7921	112.13745 59470 64143
49	8.52516 13610 65414	8192	112.83060 31276 24088
64	9.21830 85416 25360	9409	117.40531 41061 27471
81	10.31692 08302 93469	10201	122.02043 46229 68731
121	12.71481 61030 91840	10609	126.65516 36111 98366
125	14.32425 40155 25940	11449	131.32799 24456 60272
128	15.01740 11960 85886	11881	136.01934 03278 89416
169	17.58235 05535 47422	12167	139.15483 45438 18566
243	18.68096 28422 15532	12769	143.88222 23625 30906
256	19.37411 00227 75477	14641	146.28011 76353 29277
289	22.20732 33668 31693	15625	147.88955 55477 63377
343	24.15323 35158 87007	16129	152.73374 26342 21969
361	27.09767 24950 53447	16384	153.42688 98147 81914
512	27.79081 96756 13392	16807	155.37279 99638 37227
529	30.92631 38915 42542	17161	160.24799 72870 38379
625	32.53575 18039 76642	18769	165.16797 82128 66504
729	33.63436 40926 44752	19321	170.10245 21459 97195
841	37.00165 99226 31226	19683	171.20106 44346 65305
961	40.43564 71271 16372	22201	176.20501 07406 10764
1024	41.12879 43076 76318	22801	181.22229 05774 25689
1331	43.52668 95804 74688	24389	184.58958 64074 12163
1369	47.13760 74931 18913	24649	189.64583 22127 60471
1681	50.85117 95598 23221	26569	194.73958 24135 67233
1849	54.61237 96755 16783	27889	199.85757 62259 83988
2048	55.30552 68560 76728	28561	202.42252 55834 45525
2187	56.40413 91447 44838	29791	205.85651 27879 30671
2197	58.96908 85022 06375	29929	211.00980 43824 28450
2209	62.81923 61039 16433	32041	216.19719 01882 69205
2401	64.76514 62529 71747	32761	221.39568 72195 35031
2809	68.73543 81665 23868	32768	222.08883 44000 94976
3125	70.34487 60789 57969	36481	227.34110 78281 41606
3481	74.42241 35228 63688	37249	232.60379 80170 46492
3721	78.53328 73870 37000	38809	237.88700 17457 84480
4096	79.22643 45675 96945	39601	243.18030 65705 08972
4489	83.43112 71869 87911	44521	248.53216 47039 85039
4913	86.26434 05310 44127	49729	253.93933 64754 45158

TABLE IV

n	$-\zeta'(n)$	$-\zeta'(n)/\zeta(n)$	$\sum_p p^{-n} \log p$
2	0.93754 82543 15843 75	0.56996 09930 94532 80	0.49309 11093 68764 43
3	0.19812 62428 85636 85	0.16482 26821 58277 24	0.15075 75555 43950 43
4	6891 12658 96125 38	6366 97649 55371 13	6060 76333 50770 08
5	2857 37805 09462 95	2755 61921 91530 47	2683 86012 76798 36
6	1285 21651 31795 72	1263 30690 32511 06	1245 90807 22800 00
7	603 35169 60875 64	598 35585 70638 40	594 06890 39148 20
8	290 19525 53710 67	289 01683 08046 75	287 95247 08729 24
9	141 59822 27241 81	141 31440 78811 70	141 04919 21424 53
10	69 70330 08171 39	69 63404 45284 02	69 56784 47344 62
11	34 50222 22368 36	34 48518 00538 42	34 46864 25630 50
12	17 13828 45854 35	17 13406 81216 67	17 12993 52446 21
13	8 53239 08655 93	8 53134 39558 17	8 53031 09167 11
14	4 25414 93381 78	4 25388 87954 23	4 25363 05574 13
15	2 12310 85533 00	2 12304 36131 40	2 12297 90562 75
16	1 06024 42032 51	1 06022 80005 72	1 06021 18616 76
17	52968 83357 53	52968 42904 49	52968 02557 64
18	26470 02978 82	26469 92874 47	26469 82787 80
19	13230 23694 78	13230 21170 17	13230 18648 51
20	6613 53020 74	6613 52389 83	6613 51759 42
21	3306 23676 77	3306 23519 08	3306 23361 48
22	1652 94254 16	1652 94214 75	1652 94175 35
23	826 41272 38	826 41262 53	826 41252 68
24	413 18686 29	413 18683 83	413 18681 37
25	206 58693 60	206 58692 98	206 58692 36
26	103 29130 38	103 29130 23	103 29130 08
27	51 64493 08	51 64493 04	51 64493 00
28	25 82222 51	25 82222 50	25 82222 49
29	12 91103 25	12 91103 24	12 91103 24

BIBLIOGRAPHY

1. C. L. BAKER AND F. J. GRUENBERGER, *The first six million prime numbers*, Microcard Foundation, West Salem, Wisconsin, 1959.
2. H. T. DAVIS, *Tables of the higher mathematical functions, Vol. II*, Bloomington, Indiana, The Principia Press, Inc., 1935.
3. P. FINSLER, *Über die Primzahlen zwischen n und 2n*, Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, Zürich, Orell-Füssli, 1945, pp. 118–122.
4. J. W. L. GLAISHER, *Table of the values of* $\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdots\frac{x-1}{x}$, *the denominators being the series of prime numbers*, Messenger of Mathematics, vol. 28 (1898), pp. 1–17.
5. J. P. GRAM, *Undersøgelser angaaende Maengden af Primtal under en given Graense*, K. Danske Vidensk. Selskabs Skrifter, Naturv. og Math. Afd., ser. 6, vol. 2 (1881–86), pp. 183–308.
6. A. E. INGHAM, *The distribution of prime numbers*, Cambridge Tract No. 30, Cambridge University Press, 1932.
7. E. LANDAU, *Handbuch der Lehre von der Verteilung der Primzahlen*, 2 vols., Leipzig, Teubner, 1909. Reprinted in 1953 by Chelsea Publishing Co., New York.
8. D. H. LEHMER, *On the roots of the Riemann zeta-function*, Acta Math., vol. 95 (1956), pp. 291–298.
9. ———, *Extended computation of the Riemann zeta-function*, Mathematika, vol. 3 (1956), pp. 102–108.
10. D. N. LEHMER, *List of prime numbers from* 1 *to* 10,006,721, Carnegie Institution of Washington, Publication No. 165, 1914.
11. BARKLEY ROSSER, *The n-th prime is greater than n* log *n*, Proc Lond. Math. Soc. (2), vol. 45 (1939), pp. 21–44.
12. ———, *Explicit bounds for some functions of prime numbers*, Amer. J. Math., vol. 63 (1941), pp. 211–232.
13. J. BARKLEY ROSSER, *Explicit remainder terms for some asymptotic series*, Journal of Rational Mechanics and Analysis (J. Math. Mech.), vol. 4 (1955), pp. 595–626.
14. P. TCHEBICHEF, *Mémoire sur les nombres premiers*, J. Math. Pures Appl. (1), vol. 17 (1852), pp. 366–390.
15. E. TROST, *Primzahlen*, Basel, Birkhäuser, 1953.
16. A. WALTHER, *Anschauliches zur Riemannschen Zetafunktion*, Acta Math., vol. 48 (1926), pp. 393–400.
17. I. M. VINOGRADOV, *Novaya ocenka funkcii* $\zeta(1 + it)$, Izv. Akad. Nauk SSSR, Ser. Mat., vol. 22 (1958), pp. 161–164.
18. *Tables of sine, cosine and exponential integrals*, 2 vols., W.P.A. Tables prepared in 1940 under the sponsorship of the National Bureau of Standards.
19. KENNETH I. APPEL AND J. BARKLEY ROSSER, *Table for estimating functions of primes*, IDA-CRD Technical Report Number 4, September, 1961 (available on request from Communications Research Division, Institute for Defense Analyses, von Neumann Hall, Princeton, New Jersey).

INSTITUTE FOR DEFENSE ANALYSES
PRINCETON, NEW JERSEY
PENNSYLVANIA STATE UNIVERSITY
UNIVERSITY PARK, PENNSYLVANIA