SOME EIGENFUNCTION FORMULAE

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1. THE object of this paper is to correct § 4.14 and § 5.8 of *Eigenfunction Expansions associated with Second-order Differential Equations* by E. C. Titchmarsh. The example there given is inconsistent with the general theorem, and actually both of them are incorrect.

It is a question of the spectrum associated with the differential equation

$$\frac{d^2\phi}{dx^2}+\{\lambda-q(x)\}\phi = 0 \quad (0 \leqslant x < \infty) \tag{1.1}$$

in the case where $q'(x) < 0$, $q(x) \to -\infty$,

$$q'(x) = O\{|q(x)|^c\} \quad (0 < c < \tfrac{3}{2}),$$

$q''(x)$ is ultimately of one sign, and

$$\int^{\infty} |q(x)|^{-\frac{1}{2}}\, dx \tag{1.2}$$

is convergent. (The condition $q(x) \leqslant 0$ imposed is not really relevant.) The analysis given in § 5.8 is correct so far as positive values of λ are concerned, and the result stated, that the spectrum is discrete for $\lambda > 0$, is correct. *The actual result, however, is that under the above conditions the whole spectrum is discrete* (and not that it is continuous in $(-\infty, 0)$, as is stated in the book). The mistake arises in the first formula on p. 109. This contains the function

$$\xi(t) = \xi(t,\lambda) = \int_0^t \{\lambda-q(x)\}^{\frac{1}{2}}\, dx,$$

which may have a branch-point at $\lambda = q(0)$. The conclusion that $\mu(\lambda)$ is an integral function is therefore false, and the argument based on it fails.

The result will be proved by an appropriate modification of the method used in the book, although the proof may be shortened by appealing to a theorem of Weyl,† which is to the effect that the spectrum associated with (1.1) and certain boundary conditions is discrete when the equation is of limit-circle type.

The main mistake in § 4.14 is that the discrete spectrum arising from the zeros of $\sin(i\pi\sqrt{\lambda})$ is ignored.

† H. Weyl, *Math. Annalen*, 68 (1910), 220–69, § 8; also E. C. Titchmarsh, *Quart. J. of Math.* (Oxford), 12 (1941), 33–50, § 9.

2. Let $\phi(x) = \phi(x,\lambda)$ be the solution of (1.1) which satisfies the boundary conditions

$$\phi(0) = \sin\alpha, \qquad \phi'(0) = -\cos\alpha, \tag{2.1}$$

where α is a given real number. Then $\phi(x,\lambda)$ is an integral function of λ, for each x.

As in the book, let

$$\xi(x,\lambda) = \int_0^x \{\lambda-q(t)\}^{\frac{1}{2}}\,dt,$$

let
$$R(x,\lambda) = -\frac{q''(x)}{4\{\lambda-q(x)\}^{\frac{3}{2}}}-\frac{5}{16}\frac{q'^2(x)}{\{\lambda-q(x)\}^{\frac{5}{2}}},$$

and let
$$p(x,\lambda) = \{\lambda-q(x)\}^{-\frac{1}{4}}.$$

Let ρ be a real constant chosen so that $\rho-q(x) > 0$ for $0 \leqslant x < \infty$. Then

$$\int_0^x p(t,\rho)\{q(t)-\lambda\}\phi(t)\sin\{\xi(x,\rho)-\xi(t,\rho)\}\,dt$$
$$= \int_0^x p(t,\rho)\phi''(t)\sin\{\xi(x,\rho)-\xi(t,\rho)\}\,dt$$
$$= -p(0,\rho)\phi'(0)\sin\xi(x,\rho)-\int_0^x \phi'(t)[p'(t,\rho)\sin\{\xi(x,\rho)-\xi(t,\rho)\}-$$
$$-p(t,\rho)\{\rho-q(t)\}^{\frac{1}{2}}\cos\{\xi(x,\rho)-\xi(t,\rho)\}]\,dt$$
$$= p(0,\rho)\cos\alpha\sin\xi(x,\rho)+\phi(x)\{\rho-q(x)\}^{\frac{1}{4}}+$$
$$+\sin\alpha\, p'(0,\rho)\sin\xi(x,\rho)-\sin\alpha\{\rho-q(0)\}^{\frac{1}{4}}\cos\xi(x,\rho)+$$
$$+\int_0^x \phi(t)[p''(t,\rho)-p(t,\rho)\{\rho-q(t)\}]\sin\{\xi(x,\rho)-\xi(t,\rho)\}\,dt-$$
$$-\int_0^x \phi(t)\left[2p'(t,\rho)\{\rho-q(t)\}^{\frac{1}{2}}-\tfrac{1}{2}p(t,\rho)\frac{q'(t)}{\{\rho-q(t)\}^{\frac{1}{2}}}\right]\cos\{\xi(x,\rho)-\xi(t,\rho)\}\,dt.$$

Hence

$$\phi(x)\{\rho-q(x)\}^{\frac{1}{4}} = -\left\{\frac{\cos\alpha}{\{\rho-q(0)\}^{\frac{1}{4}}}+\frac{q'(0)\sin\alpha}{4\{\rho-q(0)\}^{\frac{5}{4}}}\right\}\sin\xi(x,\rho)+$$
$$+\sin\alpha\{\rho-q(0)\}^{\frac{1}{4}}\cos\xi(x,\rho)+$$
$$+\int_0^x \phi(t)\{\rho-q(t)\}^{\frac{1}{4}}\left\{\frac{\rho-\lambda}{\{\rho-q(t)\}^{\frac{1}{2}}}+R(t,\rho)\right\}\sin\{\xi(x,\rho)-\xi(t,\rho)\}\,dt. \tag{2.2}$$

If $\rho = \lambda$, this reduces to formula (5.4.5) of the book.

Writing
$$\phi_1(x) = \phi(x)\{\rho-q(x)\}^{\frac{1}{4}},$$
$$R_1(t) = \frac{\rho-\lambda}{\{\rho-q(t)\}^{\frac{1}{2}}}+R(t,\rho),$$
we have
$$\int_0^\infty |R_1(t)|\,dt < \infty.$$

Then, as in § 5.5, (2.2) is of the form
$$\phi_1(x) = A\cos\xi(x,\rho)+B\sin\xi(x,\rho)+ \\ +\int_0^x \phi_1(t)R_1(t)\sin\{\xi(x,\rho)-\xi(t,\rho)\}\,dt. \quad (2.3)$$

It follows, as in § 5.6, firstly that $\phi_1(x)$ is bounded (uniformly in any finite λ-region), and secondly that, as $x\to\infty$,
$$\phi_1(x) = A\cos\xi(x,\rho)+B\sin\xi(x,\rho)+ \\ +\int_0^\infty \phi_1(t)R_1(t)\sin\{\xi(x,\rho)-\xi(t,\rho)\}\,dt+o(1),$$
where A and B are independent of λ, and the integral converges uniformly over any finite λ-region, and so represents an integral function of λ. We have, therefore, as $x\to\infty$,
$$\phi(x)\{\rho-q(x)\}^{\frac{1}{4}} = \gamma(\lambda)\cos\xi(x,\rho)+\delta(\lambda)\sin\xi(x,\rho)+o(1), \quad (2.4)$$
where $\gamma(\lambda)$ and $\delta(\lambda)$ are integral functions of λ. (The difference between this formula and formula (5.7.3) of the book lies, of course, in the fact that here ρ is fixed, whereas in (5.7.3) $\xi(x) = \xi(x,\lambda)$ with varying λ.)

Similarly by using the differentiated form of (2.3) we obtain
$$\phi'(x)\{\rho-q(x)\}^{-\frac{1}{4}} = \delta(\lambda)\cos\xi(x,\rho)-\gamma(\lambda)\sin\xi(x,\rho)+o(1). \quad (2.5)$$

Similarly, if $\theta(x,\lambda)$ is the solution of (1.1) such that
$$\theta(0,\lambda) = \cos\alpha, \qquad \theta'(0,\lambda) = \sin\alpha,$$
we have
$$\theta(x)\{\rho-q(x)\}^{\frac{1}{4}} = \gamma_1(\lambda)\cos\xi(x,\rho)+\delta_1(\lambda)\sin\xi(x,\rho)+o(1), \quad (2.6)$$
$$\theta'(x)\{\rho-q(x)\}^{-\frac{1}{4}} = \delta_1(\lambda)\cos\xi(x,\rho)-\gamma_1(\lambda)\sin\xi(x,\rho)+o(1), \quad (2.7)$$
where $\gamma_1(\lambda)$, $\delta_1(\lambda)$ are obtained from $\gamma(\lambda)$ and $\delta(\lambda)$ by replacing $\sin\alpha$, $-\cos\alpha$, ϕ by $\cos\alpha$, $\sin\alpha$, θ respectively. Thus $\gamma_1(\lambda)$ and $\delta_1(\lambda)$ are also integral functions of λ.

We note that $\theta(x,\lambda)$ and $\phi(x,\lambda)$ are both $L^2(0,\infty)$ for all values of λ, so that we are in Weyl's limit-circle case. According to the general theory, we have to consider the limit of

$$l(\lambda) = -\frac{\theta(b,\lambda)\cot\beta+\theta'(b,\lambda)}{\phi(b,\lambda)\cot\beta+\phi'(b,\lambda)}$$

as $b \to \infty$.

Substituting (2.4)–(2.7) on the right-hand side, and putting

$$\cot\beta = \{\rho - q(b)\}^{\frac{1}{2}}\cot\beta',$$

we obtain

$$l(\lambda) = -\frac{\gamma_1(\lambda)\cos\{\xi(b,\rho)+\beta'\}+\delta_1(\lambda)\sin\{\xi(b,\rho)+\beta'\}+o(1)}{\gamma(\lambda)\cos\{\xi(b,\rho)+\beta'\}+\delta(\lambda)\sin\{\xi(b,\rho)+\beta'\}+o(1)}.$$

Choosing β' as a function of b so that $\xi(b,\rho)+\beta' = \kappa$, we obtain

$$l(\lambda) \to -\frac{\gamma_1(\lambda)\cos\kappa+\delta_1(\lambda)\sin\kappa}{\gamma(\lambda)\cos\kappa+\delta(\lambda)\sin\kappa}.$$

As κ varies this describes a circle, which is therefore the limit-circle. Thus, in the notation of the book,

$$m(\lambda) = -\frac{\gamma_1(\lambda)\cos\kappa+\delta_1(\lambda)\sin\kappa}{\gamma(\lambda)\cos\kappa+\delta(\lambda)\sin\kappa}.$$

For any κ this is a meromorphic function of λ. The eigenvalues are its poles, and so the whole spectrum is discrete.

3. As an example, we shall consider the case $q(x) = -e^{2x}$, with $\alpha = 0$ in the boundary conditions at $x = 0$. Solutions of the equation

$$\frac{d^2\phi}{dx^2}+(\lambda+e^{2x})\phi = 0 \tag{3.1}$$

are $J_\nu(e^x)$, $J_{-\nu}(e^x)$, where $\nu = i\surd\lambda$. Since

$$J_\nu(z)J'_{-\nu}(z)-J'_\nu(z)J_{-\nu}(z) = -\frac{2\sin\nu\pi}{\pi z}, \tag{3.2}$$

we have

$$\theta(x,\lambda) = -\frac{\pi}{2\sin\nu\pi}\{J_\nu(e^x)J'_{-\nu}(1)-J_{-\nu}(e^x)J'_\nu(1)\}, \tag{3.3}$$

$$\phi(x,\lambda) = -\frac{\pi}{2\sin\nu\pi}\{J_\nu(e^x)J_{-\nu}(1)-J_{-\nu}(e^x)J_\nu(1)\}. \tag{3.4}$$

We can take $\rho = 0$, so that

$$\xi(x,\rho) = \xi(x,0) = \int_0^x e^t\,dt = e^x-1.$$

Thus the general theory gives

$$e^{\frac{1}{2}x}\phi(x,\lambda) = \gamma(\lambda)\cos(e^x-1)+\delta(\lambda)\sin(e^x-1)+o(1),$$

$$e^{\frac{1}{2}x}\theta(x,\lambda) = \gamma_1(\lambda)\cos(e^x-1)+\delta_1(\lambda)\sin(e^x-1)+o(1).$$

In fact it is known that

$$J_\nu(z) \sim \sqrt{\left(\frac{2}{\pi z}\right)}\cos(z-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi).$$

Hence

$$\phi(x) \sim -\frac{\sqrt{\pi}}{\sqrt{2}\sin\nu\pi}\, e^{-\frac{1}{2}x}\{J_{-\nu}(1)\cos(e^x-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)-J_\nu(1)\cos(e^x+\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)\}.$$

Comparing the two formulae, we obtain

$$\gamma(\lambda) = -\frac{\sqrt{\pi}}{\sqrt{2}\sin\nu\pi}\{J_{-\nu}(1)\cos(\tfrac{1}{2}\nu\pi+\tfrac{1}{4}\pi-1)-J_\nu(1)\cos(\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi+1)\}, \quad (3.5)$$

$$\delta(\lambda) = -\frac{\sqrt{\pi}}{\sqrt{2}\sin\nu\pi}\{J_{-\nu}(1)\sin(\tfrac{1}{2}\nu\pi+\tfrac{1}{4}\pi-1)+J_\nu(1)\sin(\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi+1)\}, \quad (3.6)$$

and similarly for $\gamma_1(\lambda)$ and $\delta_1(\lambda)$ with $J'_\nu(1)$, $J'_{-\nu}(1)$ replacing $J_\nu(1)$, $J_{-\nu}(1)$. Hence the denominator in $m(\lambda)$ is

$$\begin{aligned}&\{J_{-\nu}(1)\cos(\tfrac{1}{2}\nu\pi+\tfrac{1}{4}\pi-1)-J_\nu(1)\cos(\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi+1)\}\cos\kappa+\\ &\quad+\{J_{-\nu}(1)\sin(\tfrac{1}{2}\nu\pi+\tfrac{1}{4}\pi-1)+J_\nu(1)\sin(\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi+1)\}\sin\kappa\\ &\qquad= J_{-\nu}(1)\cos(\tfrac{1}{2}\nu\pi+\tfrac{1}{4}\pi-1-\kappa)-J_\nu(1)\cos(\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi+1+\kappa),\end{aligned}$$

and similarly for the numerator. Thus

$$m(\lambda) = -\frac{J'_{-\nu}(1)\cos(\tfrac{1}{2}\nu\pi+\tfrac{1}{4}\pi-1-\kappa)-J'_\nu(1)\cos(\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi+1+\kappa)}{J_{-\nu}(1)\cos(\tfrac{1}{2}\nu\pi+\tfrac{1}{4}\pi-1-\kappa)-J_\nu(1)\cos(\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi+1+\kappa)}. \quad (3.7)$$

Taking e.g. $1+\kappa = \frac{1}{4}\pi$, we obtain

$$m(\lambda) = -\frac{J'_{-\nu}(1)-J'_\nu(1)}{J_{-\nu}(1)-J_\nu(1)}. \quad (3.8)$$

The corresponding function

$$\psi(x,\lambda) = \theta(x,\lambda)+m(\lambda)\phi(x,\lambda)$$

is

$$\psi(x,\lambda) = \frac{J_\nu(e^x)-J_{-\nu}(e^x)}{J_\nu(1)-J_{-\nu}(1)} \quad (3.9)$$

and the expansion formula is easily obtained. Similarly, if $1+\kappa = \frac{3}{4}\pi$, then

$$m(\lambda) = -\frac{J'_\nu(1)+J'_{-\nu}(1)}{J_\nu(1)+J_{-\nu}(1)} \quad (3.10)$$

and

$$\psi(x,\lambda) = \frac{J_\nu(e^x)+J_{-\nu}(e^x)}{J_\nu(1)+J_{-\nu}(1)}. \quad (3.11)$$

4. Consider next the case where $q(x) = -e^{2x}$, and where the interval is $-\infty < x < \infty$. If we were considering the interval $-\infty < x < 0$ only, we should obtain a continuous spectrum in $\lambda > 0$, by the theory of § 5.3, since $|q(x)|$ is integrable over $(-\infty, 0)$. It follows that, in the case of the whole interval $-\infty < x < \infty$, the spectrum will be discrete in $\lambda < 0$ and continuous in $\lambda > 0$. This we shall now verify directly.

Let $\theta(x,\lambda)$, $\phi(x,\lambda)$ denote the same functions as in § 3, and let

$$\psi_1(x,\lambda) = \theta(x,\lambda)+m_1(\lambda)\phi(x,\lambda),$$
$$\psi_2(x,\lambda) = \theta(x,\lambda)+m_2(\lambda)\phi(x,\lambda)$$

be solutions of (3.1) which are $L^2(-\infty,0)$ and $L^2(0,\infty)$ respectively. Now as $x \to -\infty$, $e^x \to 0$, and

$$J_\nu(e^x) \sim \frac{(\frac{1}{2}e^x)^\nu}{\Gamma(\nu+1)} = \frac{e^{ix\sqrt{\lambda}}}{2^\nu\Gamma(\nu+1)}.$$

If $\sqrt{\lambda} = \sigma+it$ $(\sigma > 0,\ t > 0)$, then $|e^{ix\sqrt{\lambda}}| = e^{-xt}$, so that $J_\nu(e^x)$ is not $L^2(-\infty,0)$; but $J_{-\nu}(e^x)$ is $L^2(-\infty,0)$. Hence we must have

$$m_1(\lambda) = -\frac{J'_{-\nu}(1)}{J_{-\nu}(1)}, \tag{4.1}$$

and so

$$\psi_1(x,\lambda) = \frac{J_{-\nu}(e^x)}{J_{-\nu}(1)}. \tag{4.2}$$

The interval $(0,\infty)$ gives the limit-circle case as in § 3, and so involves an arbitrary parameter. We shall consider the two particular cases referred to in § 3. We may first take $m_2(\lambda)$ to be defined by (3.8), and $\psi_2(x,\lambda)$ by (3.9). The expansion formula will then be obtained by integrating with respect to λ the function

$$\Phi(x,\lambda) = \frac{\psi_2(x,\lambda)}{m_1(\lambda)-m_2(\lambda)}\int\limits_{-\infty}^{x}\psi_1(\xi,\lambda)f(\xi)\,d\xi+$$
$$+\frac{\psi_1(x,\lambda)}{m_1(\lambda)-m_2(\lambda)}\int\limits_{x}^{\infty}\psi_2(\xi,\lambda)f(\xi)\,d\xi$$
$$= \frac{\pi}{2\sin\nu\pi}\{J_\nu(e^x)-J_{-\nu}(e^x)\}\int\limits_{-\infty}^{x}J_{-\nu}(e^\xi)f(\xi)\,d\xi+$$
$$+\frac{\pi}{2\sin\nu\pi}J_{-\nu}(e^x)\int\limits_{x}^{\infty}\{J_\nu(e^\xi)-J_{-\nu}(e^\xi)\}f(\xi)\,d\xi. \tag{4.3}$$

This has a branch-point at $\lambda = 0$. The factor $\sin\nu\pi = \sin(i\sqrt{\lambda}\pi)$ vanishes at the points $\lambda = -n^2$ $(n = 1,2,\ldots)$, but owing to the relation

$$J_{-n}(z) = (-1)^n J_n(z)$$

there is a pole only if n is odd, say for $n = 2m+1$ $(m = 0,1,\ldots)$. The residue is

$$(4m+2)J_{2m+1}(e^x)\int\limits_{-\infty}^{\infty}J_{2m+1}(e^\xi)f(\xi)\,d\xi.$$

Allowing for these residues, the integral

$$\frac{1}{2\pi i}\int \Phi(x,\lambda)\,d\lambda,$$

taken round a large circle in the positive direction, may be reduced to an integral round a loop starting at $+\infty$, encircling the origin in the positive direction, and returning to $+\infty$. The contribution to this of the first term in (4.3) is

$$-\frac{1}{4i}\int_0^\infty \frac{J_\nu(e^x)-J_{-\nu}(e^x)}{\sin\nu\pi}\,d\lambda\int_{-\infty}^x J_{-\nu}(e^\xi)f(\xi)\,d\xi+$$
$$+\frac{1}{4i}\int_0^\infty \frac{J_{-\nu}(e^x)-J_\nu(e^x)}{\sin(-\nu\pi)}\,d\lambda\int_{-\infty}^x J_\nu(e^\xi)f(\xi)\,d\xi$$
$$=\frac{1}{4i}\int_0^\infty \frac{J_\nu(e^x)-J_{-\nu}(e^x)}{\sin\nu\pi}\,d\lambda\int_{-\infty}^x [J_\nu(e^\xi)-J_{-\nu}(e^\xi)]f(\xi)\,d\xi$$
$$=-\frac{1}{4}\int_0^\infty \frac{J_{i\surd\lambda}(e^x)-J_{-i\surd\lambda}(e^x)}{\sinh\pi\surd\lambda}\,d\lambda\int_{-\infty}^x [J_{i\surd\lambda}(e^\xi)-J_{-i\surd\lambda}(e^\xi)]f(\xi)\,d\xi,$$

and similarly for the other part.

Hence the complete expansion formula is

$$f(x)=\sum_{m=0}^\infty (4m+2)J_{2m+1}(e^x)\int_{-\infty}^\infty J_{2m+1}(e^\xi)f(\xi)\,d\xi-$$
$$-\int_0^\infty \frac{J_{i\surd\lambda}(e^x)-J_{-i\surd\lambda}(e^x)}{4\sinh\pi\surd\lambda}\,d\lambda\int_{-\infty}^\infty [J_{i\surd\lambda}(e^\xi)-J_{-i\surd\lambda}(e^\xi)]f(\xi)\,d\xi. \quad (4.4)$$

Similarly, if $m_2(\lambda)$ is defined by (3.10), and $\psi_2(x,\lambda)$ by (3.11), we obtain

$$\Phi(x,\lambda)=\frac{\pi}{2\sin\nu\pi}\{J_\nu(e^x)+J_{-\nu}(e^x)\}\int_{-\infty}^x J_{-\nu}(e^\xi)f(\xi)\,d\xi+$$
$$+\frac{\pi}{2\sin\nu\pi}J_{-\nu}(e^x)\int_x^\infty \{J_\nu(e^\xi)+J_{-\nu}(e^\xi)\}f(\xi)\,d\xi,$$

and the expansion formula is

$$f(x)=\sum_{m=1}^\infty 4mJ_{2m}(e^x)\int_{-\infty}^\infty J_{2m}(e^\xi)f(\xi)\,d\xi+$$
$$+\int_0^\infty \frac{J_{i\surd\lambda}(e^x)+J_{-i\surd\lambda}(e^x)}{4\sinh\pi\surd\lambda}\,d\lambda\int_{-\infty}^\infty [J_{i\surd\lambda}(e^\xi)+J_{-i\surd\lambda}(e^\xi)]f(\xi)\,d\xi. \quad (4.5)$$

If we put $e^x = s$, $e^\xi = t$, and

$$f(x) = f(\log s) = g(s),$$

these formulae take the form

$$g(s) = \sum_{m=0}^{\infty} (4m+2)J_{2m+1}(s) \int_0^\infty J_{2m+1}(t)g(t)\frac{dt}{t} - \\ - \int_0^\infty \frac{J_{i\surd\lambda}(s)-J_{-i\surd\lambda}(s)}{4\sinh\pi\surd\lambda}\,d\lambda \int_0^\infty [J_{i\surd\lambda}(t)-J_{-i\surd\lambda}(t)]g(t)\frac{dt}{t} \tag{4.6}$$

and

$$g(s) = \sum_{m=1}^{\infty} 4mJ_{2m}(s) \int_0^\infty J_{2m}(t)g(t)\frac{dt}{t} + \\ + \int_0^\infty \frac{J_{i\surd\lambda}(s)+J_{-i\surd\lambda}(s)}{4\sinh\pi\surd\lambda}\,d\lambda \int_0^\infty [J_{i\surd\lambda}(t)+J_{-i\surd\lambda}(t)]g(t)\frac{dt}{t}. \tag{4.7}$$

5. We shall discuss now the relation between the above formulae and the Webb–Kapteyn theory of Neumann series.†

A Neumann series for an odd function $g(s)$ is of the form

$$\sum_{m=0}^{\infty} a_{2m+1} J_{2m+1}(s), \tag{5.1}$$

and it is known that certain classes of analytic functions can be expanded in this form. However, in view of the formula

$$\int_0^\infty J_{2m+1}(t)J_{2n+1}(t)\frac{dt}{t} = \begin{cases} 0 & (m \neq n), \\ 1/(4n+2) & (m = n) \end{cases} \tag{5.2}$$

it is also possible to calculate the coefficients a_{2m+1} in the case of an arbitrary function of a real variable $g(s)$. Proceeding in the manner of an ordinary Fourier series, we obtain

$$a_{2m+1} = (4m+2) \int_0^\infty g(t)J_{2m+1}(t)\frac{dt}{t}. \tag{5.3}$$

It was shown, however, by Kapteyn, that the series (5.1), with the coefficients (5.3), can represent the function $g(s)$ only if $g(s)$ satisfies the integral equation

$$g'(s) = \frac{1}{2} \int_0^\infty \frac{J_1(t)}{t}\{g(t+s)+g(t-s)\}\,dt. \tag{5.4}$$

† G. N. Watson, *Theory of Bessel Functions*, § 16.4.

For an odd function $g(s)$, this is equivalent to

$$g'(s) = \frac{1}{2}\int\limits_0^\infty \frac{J_1(t)}{t}\{g(s+t)-g(s-t)\}\,dt. \tag{5.5}$$

This equation was solved by Hardy and Titchmarsh.† It was shown, for example, that, if $g(s)$ is of integrable square over $(-\infty,\infty)$, then a necessary and sufficient condition for (5.5) to be satisfied is that $g(s)$ should be of the form

$$g(s) = \int\limits_0^1 \{a(u)\cos su + b(u)\sin su\}\,du, \tag{5.6}$$

where $a(u)$ and $b(u)$ are $L^2(0,1)$. In particular it follows that $g(s)$ must be an integral function of s of a certain type.

The formulae of § 4 throw a new light on all these questions. The Webb–Kapteyn Neumann expansion of $g(s)$ is just the formula (4.6) without the repeated-integral term. The condition that it should be valid is that the continuous spectrum should make no contribution to the expansion. It is to be expected that this will hold only for special classes of functions.

To prove the sufficiency of the Hardy–Titchmarsh condition directly, suppose that $g(s)$ is of the form (5.6) and is odd, so that in fact it is of the form

$$g(s) = \int\limits_0^1 b(u)\sin su\,du, \tag{5.7}$$

where $b(u)$ is $L^2(0,1)$. Then

$$\int\limits_0^\infty \{J_{i\surd\lambda}(t)-J_{-i\surd\lambda}(t)\}g(t)\frac{dt}{t} = \int\limits_0^1 b(u)\,du \int\limits_0^\infty \{J_{i\surd\lambda}(t)-J_{-i\surd\lambda}(t)\}\frac{\sin tu}{t}\,dt, \tag{5.8}$$

the inversion being justified by absolute convergence. Now,‡ if $0 < u < 1$,

$$\int\limits_0^\infty \frac{J_\nu(t)\sin tu}{t}\,dt = \frac{\sin(\nu \arcsin u)}{\nu}.$$

Hence the inner integral in (5.8) vanishes, and the result follows.

To prove a theorem of the converse type, we observe that the Parseval

† G. H. Hardy and E. C. Titchmarsh, *Proc. London Math. Soc.* (2), 23 (1923), 1–26.

‡ G. N. Watson, loc. cit. § 13.42 (2).

formula corresponding to (4.6) is

$$\int_0^\infty \frac{\{g(s)\}^2}{s}\,ds = \sum_{m=0}^{\infty}(4m+2)\left\{\int_0^\infty J_{2m+1}(t)g(t)\frac{dt}{t}\right\}^2 - \\ -\int_0^\infty \frac{1}{4\sinh\pi\surd\lambda}\left\{\int_0^\infty [J_{i\surd\lambda}(t)-J_{-i\surd\lambda}(t)]g(t)\frac{dt}{t}\right\}^2 d\lambda.$$

For the contribution of the continuous spectrum to vanish, the second term on the right must vanish, and therefore

$$\int_0^\infty [J_{i\surd\lambda}(t)-J_{-i\surd\lambda}(t)]\frac{g(t)}{t}\,dt = 0 \tag{5.9}$$

for all positive values of λ.

In order to avoid intricate analysis, we shall assume in this problem that $g(t)$ is absolutely continuous, that $g(t)$ and $g'(t)$ are both $L^2(0,\infty)$, and that $g(0) = 0$. As in Theorem 68 of Titchmarsh's *Fourier Integrals*, it follows that, if $\surd(\frac{1}{2}\pi)\,b(u)$ is the sine transform of $g(t)$, then $b(u)$ and $ub(u)$ are both $L^2(0,\infty)$.

Since $J_\nu(t)/t$ is $L^2(0,\infty)$ if $\mathbf{R}(\nu) > \frac{1}{2}$, we can calculate

$$\int_0^\infty \frac{J_\nu(t)}{t}g(t)\,dt$$

by the formulae of Fourier sine-transforms. We have

$$\int_0^\infty \frac{J_\nu(t)}{t}\sin ut\,dt = \begin{cases} \nu^{-1}\sin(\nu\arcsin u) & (u<1), \\ \dfrac{\sin\frac{1}{2}\nu\pi}{\nu[u+\surd(u^2-1)]^\nu} & (u>1). \end{cases}$$

Hence, if $\mathbf{R}(\nu) > \frac{1}{2}$,

$$\int_0^\infty \frac{J_\nu(t)}{t}g(t)\,dt = \int_0^1 \frac{\sin(\nu\arcsin u)}{\nu}b(u)\,du + \int_1^\infty \frac{\sin\frac{1}{2}\nu\pi}{\nu[u+\surd(u^2-1)]^\nu}b(u)\,du.$$

Under the above conditions, these integrals converge uniformly with respect to ν for $\mathbf{R}(\nu) > -\frac{1}{2}$, $|\nu| \geqslant \delta > 0$, so that the formula holds, by the theory of analytic continuation, in this region. In particular it holds if ν is purely imaginary. Taking $\nu = i\surd\lambda$, then $\nu = -i\surd\lambda$, and subtracting, we obtain, by (5.9),

$$\int_1^\infty \{[u+\surd(u^2-1)]^{i\surd\lambda}-[u+\surd(u^2-1)]^{-i\surd\lambda}\}b(u)\,du = 0;$$

or, writing $u = \cosh w$,

$$\int_0^\infty b(\cosh w)\sinh w \sin(w\sqrt{\lambda})\, dw = 0.$$

Since $$\int_0^\infty \{b(\cosh w)\sinh w\}^2\, dw = \int_1^\infty b^2(u)\sqrt{(u^2-1)}\, du < \infty$$

under the conditions assumed, it follows that

$$b(\cosh w)\sinh w = 0$$

for almost all values of w, i.e. that $b(u) = 0$ for almost all values of u greater than 1. Hence $g(s)$ is of the form (5.7).