Imagination and infinity

Alain Connes and Alain Prochiantz

Imaginations, a series of interviews proposed by Alain Prochiantz, neurobiologist, professor at the Collège de France.

ALAIN PROCHIANTZ : We are in the summer grid, on the program Imaginations , with philosophers, sociologists, and scholars, and artists, on the probably most unifying theme, between scientists and artists. It must be said that this is how we thought in any case thing. And today I have the pleasure of welcoming Alain Connes. So, Alain Connes is a very great mathematician. He is a professor at the College of France, where he held the Analysis and Geometry chair. He is a recipient of the greatest reward you can have in math, which is the Fields medal. And he has this interest not only in mathematics, of course, but also for music and for art, which makes it really one of the people who can make the very strong link today between art and science on a mode which is not a flat mode but which is a mode which intellectually engages those who make art or those who make science, that is to say, a real reflection on the subject of art and on the subject of science. He is a specialist in what is called non-commutative geometry and if I say that, it's probably because it's not unrelated to his interest in time, and through interest in time, interest in musical creation.

Alain, I have the duty to try to extract from you, not everything because it's inexhaustible, but in any case elements for reflection on this question of science, math, beauty in math, and its connection with artistic beauty.

ALAIN CONNES : Yes. In fact, therefore, I thought a little bit about what is simply the imagination. And the first thing that struck me was that finally, radio, as a means of communication, is a means which is much more interesting, in terms of the listener's concentration, level of listening precisely, that another means of communication like television. Why ? Because in the term

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imagination, there are the pictures, and it is extremely important that the listener does not have a purely passive role, does not receive the image as we want to impose it, but be able to create it himself, and create it from speech, from language. So this is the first thing that struck me, it was how much more appropriate a radio shows, to speak of imagination, only if we tried to illustrate it directly. The second thing that also struck me a lot is how the mathematicians have a use of the imagination which a priori is very different, very special, very different from what happens in other areas, that's just that I want to try to explain. So what I mean is that a mathematician uses imagination a lot, but he uses it in a very special way. That is to say, in fact, the role, the first role of the imagination for a mathematician, which is an essential role, is that of creating mental images.

And this role, in fact, of course, is absolutely not something passive. It's a... how to say... we can't do it only when you dry on a problem. So there is an essential virtue... for a mathematician, for example, if he is to read a book to take for example a theorem which is in a book etc. and above all, he must not to watch the demonstration, but to try to demonstrate himself. Why ? Because when he does that, in fact, he will create in his brain, I say a mental image but in fact, it's... well, it is not always something geometric, it is not always something that can be described as an image but, it's a certain assembly in his brain which then will do that when he is confronted to a page of formulas, well, that page of formulas will speak to him. And in this page of formulas, he will see actors. He will see things that will "resonate" with each other, etc. The comparison that I always want to take, is that suppose for example that you are in the metro, and that you see a subway passenger who is in the process of read a music score. When you're not a musician, this score of music tells us nothing, nothing at all. When you're not a mathematician and you see a passenger reading math formulas, you get the impression I mean that we are completely excluded and that we have no chance to understand. And in fact, the reason is precisely this fabrication of mental images and this fabrication of mental images, it involves this ability absolutely essential to imagine. This ability to imagine, there, plays an absolutely fundamental role. And I always give the following advice, I mean, for example, to a young mathematician : the advice, it is if one is confronted for example with a calculation even very difficult, of course, these are abstract calculations etc., the right method is not to rush to the computer to try to do the math, or take a spreadsheet, no ! The right method is to go for a ride on foot and try to get by with what you have, for precisely, by there reflecting, creating in the brain these mental images, and whatever complexity of the problem.

Whatever the impossible character at the start of this creation, it is in drying, and rightly, by gradually appropriating mental objects that will correspond to the problem that we will progress. So there is an active part which is absolutely essential and from my point of view, that is really the first, the fundamental role of imagination in mathematics. In this sense, it is very different from other areas, because, of course when we do physics, and when we talk about the universe, well, everyone has a mental image at the outset, of what the universe is, therefore, we are not going to need to create something out of nothing, when in math, really, we are faced with this thing. So from my point of view, there is this essential point, which is that one must be active, and one is all the more active that one is not given an already pre-established model. And if for example we were on television, we would try to illustrate concepts of mathematics by pictures but, every mathematician, every person, is a particular case and will create in his brain, a very particular image, a very very particular assembly, and it is impossible to give a generic shape of that.

Alain Prochiantz : But when you talk about mental images, I think that at some level, even outside of math, there is a moment where it is necessary to have recourse to this kind of fabrication of mental images...

Alain Connes : Absolutely.

Alain Prochiantz : of bricks that we handle...

Alain Connes : ... which fit together...

Alain Prochiantz : which interlock or which do not interlock and that, it is perhaps because precisely, mathematical thought is the only way to think, outside the formulas. It's not just formulas, it is a way of thinking which is a natural language for the scientist of a some type. So the question I wanted to ask you, to shed some little light, for me moreover, but also perhaps for those who listen to us, that is, I would say "What is the image of these mental images ? What does it look like a mental picture ?"

Alain Connes : Okay, so first, there are the most simple, of course. That is to say that if we talk about ordinary geometry, for example, plane geometry, there is a theorem that I really like, it's Morley's theorem. So there, the listener must first try to imagine a triangle. So each listener will imagine a different triangle, but whatever, okay. So we start from a triangle. And what does the theorem of Morley, what does Morley's theorem say ? Is that we cut each angle of the triangle in three equal parts, therefore in three equal angles. And we intersect the corresponding lines. We get a triangle inside, in general, triangle from which we started, and the wonder is that the triangle we get at the interior is always, always, an equilateral triangle. So this is mernight light ! This is called Morley's theorem. And when we stated this theorem, we have a mental image, of course. And in fact, the mental image that we have is much better than if we were drawing a triangle on paper, and that we have drawn the equilateral triangle in the middle. Why ? Because we would be automatically disturbed by the grain of the paper, by the pencil with which we wrote, etc. So there, there is already an abstraction that has occurred. But... there is an essential point precisely in mathematics. The essential point is that I was able to explain to you that a mental image was very simple, in the case of plane geometry. But in mathematics, we manipulates geometries which are much more complicated than that, and which in general, are geometries that have dimension much larger than 3, and even in general, of infinite dimension.

That is to say, we understood, thanks to physics for example, that the quantum mechanics, it's mechanics that occurs in a space that is called the Hilbert space, which is a space of infinite dimension. So, how is it that the mathematician can have access to this space, from infinite dimension ? This is the question, this is the essential question. And this essential question, in fact, it has a very, very, very fundamental answer. This answer is the duality between geometry and algebra. So to explain it, I'll take an example. I will take a simpler example than Morley's theorem. Suppose for example that we are struck by the fact that the medians of a triangle meet at one point. So it's okay when we do it in plane geometry. We see well what it is. We have an analogous statement when we go into geometry in the space. But what about when we look at geometry in arbitrary dimension ? It seems impossible because how are we going to represent a space of dimension 4, an object corresponding to a triangle in 4 dimensional space etc. or in a larger dimension :

the answer is wonderfully simple : the way of understanding in algebraic language, by a formula, why the medians of a triangle meet, is just to write the coordinates of what's called the barycenter of the 3 points. That is to say that we take the coordinates of the points. And then we add them up and divide it by 3, because we are in dimension two; if we were in dimension 3, we would divide by 4 and so on. So what, the wonderful thing is that when you have algebra, it's a bit like the two hemispheres of the brain, i.e. there is the right hemisphere and the left hemisphere, they communicate with each other. It is the right hemisphere which has thememental image of the triangle, as I explained to you at the beginning. That is to say, it sees it, sees Morley's triangle, etc. And then after, we communicate.

We communicate with the left hemisphere. In the left hemisphere, there is a formula. It's a formula that allows... And this formula is completely insensitive to dimension. That is, once we have it written in dimension 2, it will exist in dimension 3, in dimension 4, and even in infinite dimensions. So there is a wonder that is happening, which is that in mathematics, we are able to climb, precisely, because it is something that strengthens the mental image, that helps give it security, and that's the formula. And once we have this formula, afterwards, we will operate in another mode. And this mode is no longer the visual mode, and that's why the notion of mental image is reductive, because it reduces everything to a geometric vision. Now, in mathematics, there is a duality, between geometry and algebra. Which means on the one hand, we have this geometrical vision and there, in general, the geometrical vision, it is something that will impose itself immediately... That is to say, when you have a figure, that figure will speak to you, but it will speak to you right away.

While algebra is something entirely else. And it's something which will evolve over time, and this is something in which the calculations will be algebraically, and I admit that I, for example, am persecuted. The last night I woke up, I said to myself "in this formula, I was not mistaken?". Why? Because my brain keeps on function...

And it continues to do the calculations etc. And this is something that takes place over time, and which is not at all of the same nature, as a static mental image, a geometric image, which exists and is frozen once and for all and that we understand immediately.

Alain Prochiantz : Mental images are not necessarily static ticks. I imagine that we move them, we turn them over, we combine them.

Alain Connes : We can move them, we can return them. There is the reversal of the sphere for example.

Alain Prochiantz : But probably not just anyhow. Is there a grammar of the combination of mental images ? What is permitted in the manipulation of mental objects, of these images, and that makes it not just anything, there is a kind of grammar behind.

Alain Connes : There is of course a grammar behind it. I think that, arguably one of the most important facets of grammar is the power of analogy. And then metaphor. But I think that analogy is something extraordinarily powerful, and which for the moment, is completely inaccessible to processes like the Machine Learning, Artificial Intelligence, etc. So it's an extraordinary tool...

Alain Prochiantz : For you, it's intuition, analogy ?

Alain Connes : No, it's more than that. In other words, what happens, precisely, through mental images, is that at some given point, the brain realizes that two mental images that would appear extremely far from each other (this was what had happened to Poincaré when he got on the bus), mental images far apart of the others, I mean, he was talking about two completely different things, in fact he realized at one point that there were extraordinary similarities between the two. And the fact that there were these similarities between the two, he was able to develop an analogy afterwards between two domains which a priori have nothing to do with each other.

And so, an analogy is something that is extremely delicate to manipulate, because it's not a simple dictionary. If it was a simple dictionary, that would be razor-sharp, that is to say if one could say "such thing corresponds lay on such other thing etc., etc.". It's a kind of... There is a Japanese mathematician, whose name is Oka, who had described this wonderfully, it is a kind of transplant, a kind of... We have a little flower and very lawfully, we are trying to transplant it to another place, and why this is something incredibly fruitful and effective, it's because in general, precisely, the things that we understand on the one hand, we don't understand on the other and vice versa. So that means that we will be able to transplant the understanding that we have on one side, and see how, when we try, we make tests, we see etc. but above all it is not necessary, at that time of development, we must not try to be too rigorous, because if we are too rigorous, everything will collapse. And it is a moment, precisely, which has a poetic and artistic aspect. Why ? Because when we transplant little flowers, when we do this, if we try to be too smart, too fast, etc., we will say "well, it will not work, but it will not work for such reason etc." And then we give up, and we ruined everything.

Alain Prochiantz : It is a kind of correspondence, in fact.

Alain Connes : It's a kind of correspondence, of analogy. And we cannot try to codify it too precisely at the moment when it is find out. It is something extremely fragile and this fragility makes that, for example, when we discover it, we try to say it, we try, you have to, you have to know that mathematicians are very hard people, that is to say, in mathematics, the dream is excluded. I will come back to that. But if we have perceived an analogy between two subjects, and if we try in a way too fast, too premature, to say it, it will be destroyed. So there is a part of developing a new theory like that, in which we must protect themselves, we must protect themselves, it's like a little child who must be protected, etc. and only after a while, when it has done proofs, when it has grown enough, there we can reveal it.

Alain Prochiantz : You have to let the analogy mature, the correspondance mature, to make it strong enough, to face the test of truth.

Alain Connes : So the test of truth is something absolutely terrible. So what you need to know, when I was talking about the imagination in mathematics, naively, you would think that, imagination in mathematic is to imagine things and then try to demonstrate them, etc. But in fact, there is a yoke in mathematics, which is absolutely terrible, and which, I think, is largely similar to that of physics, but of a very different way. That is to say, in mathematics, what happens is that we can have imagination, we can imagine a new theory etc. But, the problem is that, very quickly, we will come up against a reality, which is mathematical reality, and this mathematical reality, it is terrible, in the sense that, I mean, if we don't have all the elements of a demonstration, if we don't have for example, I mean, the possibility to check things out on a computer etc. we actually realize that the freedom we enjoy is absolutely minimal. So that's why I have insisted that the role of imagination in mathematics is not to imagine new things, etc. Not at all. Is to create an image mental inside the brain. There, it is really useful. It's something essential. Against that, after, there is such a straitjacket in terms of imagination, compared to other subjects, I think of artists, I think of novelists etc., this yoke is so hard, so constraining, that in fact it prevents, it just annihilates any possibility of freedom.

Alain Prochiantz : Very good. We may be going over a first variation on a German national tune by Chopin, interpreted by Nikita Magaloff, on which you will give us a little comment.

Alain Connes : Exactly, of course, yes yes.

(Musical interlude) : On a German national tune

Alain Prochiantz : Can you tell us why you chose this track ?

Alain Connes : So, then, why did I choose this tune, these variations. Of course, for the interpretation of Nikita Magaloff, whom I like a lot. But, in fact, my reason is a very personal reason, and which has to see, how to say, with the structuring of the child's imagination. What I'm going to say, it's something very personal, so, but whatever, I think it's something generic. In fact, my two mother's grandparents come from Constantine in Algeria. They were from this city. And my childhood was lulled by the fact that my maternal grandmother was a pianist. And she was an orphan, she had become an orphan at the age of 6. Both of his parents were dead. She was in Algeria, she had been taken in a convent, and she brought me back often told stories when I was a kid, when I was very little, stories from his father. And her father, when she was very young, had offered her a piano, and he had played, on the piano, the part of Chopin's variations which is so beautiful, not the very beginning, but the major theme and that is introduced by Nikita Magaloff in an incredible way, because it's a song that might interpret it as a piece of technique but not at all, he understood exactly how much the theme exhibit should be preceded by a slowdown, etc., and how beautiful the theme is.

And in fact, my childhood was rocked by this air, which I had a lot hard to find afterwards, when I wrote the family genealogy, and so actually what I meant was I think the imaginary of a child is structured very early especially by music, and by this time, the imaginary, in the naive sense, of what the child can imagine when he hears stories like that. So I never went to Constantine. Indeed, I never went to Algeria, but I always had in my head, a extremely interesting picture, precisely, of this moment when the father of my grandmother who was a doctor, in fact, had given him this little piano. And the role it played...

Alain Prochiantz : And does that have to do with the way of thinking as a mathematician ?

Alain Connes : Well, let's say it's very well known, each mathematician is different, so I don't want to generalize. But in fact, it is very, very accepted, that in general, mathematicians are very interested in the music, failing that, necessarily, being musicians, since we don't have much of time when you're a mathematician, so if you want to practice an instrument is something that would take too long. But in general, they are very sensitive to music and it's true, it's true, and I continue what I said earlier, it's true that there is a very strong analogy between algebra and music, by the course in time...

I always explain of course the fact that language itself, language that we use all the time, is non-commutative since we cannot commutate the letters between them, unless you make anagrams but... so there is a very strong relationship actually between music and algebra, I think.

Alain Prochiantz : And therefore temporality...

Alain Connes : And temporality, of course, of course.

Alain Prochiantz : You can explain this question to us a little bit temporality and a little bit non-commutative geometry, how does that get in there ?

Alain Connes : Yes, of course. So let's say we wrote two books, with Dany Chéreau, who is my wife, and then Jacques Dixmier, who was my thesis teacher. And precisely, in these two books, we continued this theme which is an essential theme in what I have done, in my life of scientist, and that started with the discovery that when you do algebra, but in a non-commutative way, that is to say that we do not allow ourselves to swap letters between them. So of course, this is something that is essential because that's what Heisenberg found when he discovered quantum mechanics. When he discovered quantum mechanics, he understood that when deals with very small systems, microscopic systems, in fact, contra what you do when you do classical physics, where you write $e = mc^2$ or $e = c^2m$, it's the same thing. When working with a system microscopic theme, we no longer have the right to swap letters, it's extraordinary, he made a fantastic discovery. And besides, he made this discovered at 4 a.m., when he was isolated on the island of Heligoland and, what's wonderful actually, is how far discoverers get to convey, in their writings, Heisenberg did so in his memoirs, the former the traumatic vision he had when he made this discovery. And he said it was a vision that was scary, because of the fact that it was the first to see this, in fact, he had before him a landscape that was almost the whole landscape, and that was scary. And he describes it wonderfully. And he is not the only one to have been able precisely, by being the first discoverer, to transmit that.

So on my side, therefore, what I had perceived if you want, is that, in fact, when we do non-commutative geometry, automatically, it's a certain type of algebra that was discovered by Von Neumann, automatically, algebra itself secretes its own time so does that non-commutative algebra secretes time, the passage of time and this is something absolutely overwhelming in a way, and for years and years, I had been fascinated by this fact. For sure, mathematically, it has a lot of consequences because for example, algebra has periods, etc., but I had always been unable to have an idea of how this, if you will, could find its place in physics and so this is the subject of our first book, which is called The quantum theater, which is published by Odile Jacob and in which, precisely, we tried, me and my two co-authors, to transmit this find, but so that it can be seen by the public.

It's difficult, it's difficult. And in the second book then, The Specter of Atacama, we went much further in the sense that there, we tried to precisely transmit this link between forms and music, which is a link also extremely important, and that says that when we try for example to explain where we are in space, well actually, we are faced with a mathematical problem which is not at all trivial, which is not at all obvious which is the problem of being able to give a space or a form, more generally, invariantly. And what math teaches us is that if we want to give a form invariant, the first thing a form gives us is a musical range, it seems something quite surprising therefore : a form gives us a musical range called a spectrum. And after, of course, there is a whole development from there. The second book is called The Specter of Atacama because precisely what is happening, and which is quite astonishing, is that we can calculate the spectrum of ordinary forms. So if we look at a space as a drum, it's Marc Kac who had long posed the problem if you want : can we hear the shape of a drum ?

That is to say that a drum has vibrations, when you hit it, and one does not believe that you will always get the same sound ; that's what do know people who play drums for example ; so we have very different sounds but these sounds form a range, and an obvious question is "Can we recognize the shape of the drum from the scale ?".

So this is a question that has a mathematical answer but the thing is really amazing, when we can calculate the range of a geometrical object, of a geometric shape that we know, there are spectra, so I identify the range if you want with the frequency range. And those frequences, we will represent them by spectral lines. There is a problem which arises in an absolutely insolent way. In fact, there are spectra, which appear completely naturally, and here we have really a considerable difficulty, but in some cases, we get there, to find the form, the physical form, whose spectrum is the spectrum. And then there is an example that we explain and that will justify the second piece of music which I will talk about later... This is an example that we explain in large detail in the book, this is called the spectrum of the guitar.

So I'm going to try to explain it, but again, the auditor is preparing to construct mental images himself in what I will explain. So the first mental image is to imagine a guitar. Well. You have a guitar. You may have seen guitars, so you can imagine in your head what a guitar is, I don't need to show it to you. So if you look at a guitar, you'll see on the neck of the guitar, lines that are perpendicular to the neck, and which are called frets. So if you look closely at these frets, you'll see... Look at them in your head. You will see that at the start, there are no frets, there is a kind of hole, good, which will allow resonances. And then there, the frets begin. And they are not at all evenly spaced. You would have thought that simply, that when we look at the neck of the guitar, if you want, the frets will be equally spaced. In fact, they are not at all equally spaced. And the mathematician, when he sees the spacing of the frets, asks himself the question "but why did we not space the frets of equal intervals ?". So the answer is a mathematical answer, but it's an answer which is wonderful, because it will give us a spectrum. And this spectrum, afterwards, we will have to look for the form of which the spectrum is. So first, what is this spectrum ? Well, when we make music, we notice a very important thing, which is that the ear is not all sensitive to 1 2 3 4 5, etc. she is not sensitive to add up, she is sensitive in fact to multiply a frequency by something, that is to say if we take a frequency and multiply it by 2, that corresponds to the stepwise to the octave. The ear is sensitive to the passage to the octave, it feels a correspondence between the two frequencies, it feels a harmony between the two frequencies. It is the multiplication by 2. The ear is also sensitive to multiplication by 3 : when we take a frequency and when we multiply it by 3, the ear hears a resonance, it hears something that matches. So now, how does that explain the spectrum of the guitar ? It explains the spectrum of the guitar because when you raise the number 2 to the power of 19, we get practically the number 3 raised to the power 12. It can't be a tie because when we raise 2 to the power 19, we get an even number. When we raise 3 to the power 12, we get an odd number. So it can't be an equality. In fact, what happens is that if we look at the twelfth root of 2, it is a number equal to 1.05 etc., and it's practically the same thing that the 19^{th} root of 3, what does that mean ? It means that in musique, what we did, with the frets of the guitar, is that we managed to make it seem as if these two numbers were equal and the 12 in question, these are the 12 tones of the well-tempered scale. And all the music is based on it. And what is the spectrum of the guitar ? Those are the powers of the number, which is the twelfth root of 2, and which is practically the 19^{th} root of 3. So it's something extraordinary. And then, here we are faced with a problem because we have this so evident spectrum. This spectrum is before us and we wonder what is the object from which it is the spectrum. So when you're a mathematician, you have a bunch tools to watch this ? Why ? Because when we look at the spectrum which corresponds to the drum, or the spectrum which corresponds to a form which is two-dimensional, which is of dimension 2, we notice that its range, it grows like a parabola. If we looked at a 3-dimensional object, that would grow with a power of 3, etc. And then, we now look at the spectrum guitar ? *(Clicking of interrogative language)*. Ah ! He is extremely weird ! Because if we calculate its size using what I told you before, we obtain that it is an object of dimension 0, an object of dimension

0 in the sense that its dimension is smaller than any number, not zero but positive. Ah ? ! So the wonder is that in fact there is an object whose spectrum is the spectrum of the guitar, but it's an object of non-commutative geometry. So we fall back on our feet. And so in fact, the book, the book we wrote on the Specter of Atacama, this is a book which is entirely based on the fact to try to understand a spectrum. This spectrum was observed by the Observatoire of Alma in Chile and throughout the book, there is a hero, finally, there are several, there are three essential characters, there is a mathematician, there is a physicist who was there in the first book, who escaped a quantum stay and the whole book is based on trying to understand this mysterious specter in the Atacama Desert.

Alain Prochiantz : In the Atacama Desert. So let's listen now *Hi of Love*, from Elgar, played by Itzhak Perlman and after, we will resume our discussion.

Alain Connes : Of course.

(Musical interlude : Hi of Love)

Alain Prochiantz : After this *Hi of Love*, I would like to remind you that we receive today, as part of the series of interviews on the theme Imaginations, Alain Connes, mathematician and professor at the Collège de France, holder of the Analysis and Geometry chair. Alain, I think you wanted to betake precedence over this piece.

Alain Connes : Absolutely. Why did I choose this tune ? It's to illustrate exactly what I said earlier but the difference between the violin and the guitar. So the guitar, I talked about the spectrum of the guitar, and frets on the neck of the guitar. Well. It is obvious that when we have a violin, we don't have frets, and the extraordinary difficulty of the violin comes from the fact that precisely, we do not have a discrete spectrum in the mathematical sense, that is to say we do not have... If you want, an infinitesimal displacement finger on the violin neck will make all the difference. And this, in the interpretation of Itzhak Perlman, is wonderful, and that's why I wanted it to be listened to, it is wonderful because of the infinite precision it happens to have, in the sounds it produces ; mathematically, what we say, if you will, it's that the difference between the guitar and the violin is that the guitar has a discrete spectrum, which I explained earlier, the violin has a continuous spectrum. But there is in the spectrum of the violin the same difficulty as in the guitar. That is to say that there is an exponential scale, that is to say that when you go from one note to another, naively you would think that it should be done by spacing the fingers of equal length, no, it must be done by spacing of an exponential length, since that corresponds to the powers from the number before.

So in fact, there are always, always, between two violinists, infinitesimal differences and this interpretation of Itzhak Perlman is, in my point of view, a marvel because there are very small nuances, tiny, that the ear perceives of course, and that makes this adaptation wonderful.

Alain Prochiantz : Thank you very much, Alain Connes, I would have liked to come a little bit on the question of the mathematician, on the question of the demonstration in fact, because there are conjectures. How it comes a guess ? And how can we pass, afterwards, from the work on the conjecture at the demonstration of the thing and these are two different ways of doing mathematics. Are these two different types of mathematical minds ?

Alain Connes : In fact, it's always amazing what happens with conjectures. That is to say, in fact, a mathematician discovers something totally new. The example I have in mind, of course, we talk about it a lot in the book, it is Riemann, in the last century, in the XIX$^{\text{th}}$ century, more specifically, it was not the last century, that he made a discovery absolutely phenomenal. In fact, he found that we could understand the prime numbers, so understand the randomness of prime numbers, the randomness that was not controlled at all, from a function called the zeta (ζ) function and after demonstrating a formula, so it gives a formula exact if you want, for the number of prime numbers smaller than n. It's not so much the fact that there is an exact formula, because there are others, but it's the fact that this formula actually describes exactly the behavior of prime numbers. And he actually saw in this formula that he demonstrated, that there was a music of prime numbers, that is, he showed that there was a dominant term, which is easy to understand because basically the prime numbers get more and more rare, by and large, like the inverse of the number of digits of the number keep.

So when we look at prime numbers for example, between 10 000 and

100 000, or between 100 000 and 1 000 000, the proportion will be divided by 2. In fact, this thing, if you will, this phenomenon, leads to a function called the integral logarithm, which is actually the first term in Riemann's formula. But afterwards, the hazard of prime numbers manifests itself precisely by a spectrum. And it manifests itself precisely by what we could call the music of prime numbers.

So in fact, when Riemann found that, he saw himself making calculations, he did calculations, he realized that the zeros of his function, which rightly governs the spectrum, seemed to be all on a certain line. And the fact that they are on this line plays an essential role, because the fact that let them be on this line says that the formula he gave is a formula extremely precise. If there were any who were outside this line, there would be a kind of chaos which would be introduced, it would not be at all a nice thing. And he surmised that all his zeros were there. This conjecture was made roughly in the years 1850-1860, so it's been a considerable time that it was made, but, I think it was basically sure it was true, and basically he wanted to go on and make a guess is to be pretty sure that a result is true, and to go beyond.

So now, this Riemann conjecture, it has been verified with the computer, because with the computer, we can go very very far ; in fact, we have a way of calculating, which is very very efficient for this function, and we checked it for billions of zeros ; so at the verification level, we have a very strong indication. We do not know if it is true because there have been other conjectures that seemed to be true like that, but that are not true for very very large numbers, so we don't know if it is true but the way he found it was that he didn't want to stop there, if you will, and he wanted to go further.

And after him, there were a very large number of mathematicians who were interested in it, there were for example mathematicians who, when took the plane or etc., sent a letter saying "I demonstrated... etc." thinking that if the plane broke its face... , then... *(laughs)*. Here. So I have all kinds of stories, around this conjecture. But let's say that what is extraordinary is that with a conjecture like this, there is something extraordinary, it is that in fact secretly, it motivated most of the most interesting developments in mathematics in the XXth century. That is to say that if we know enough things in mathematics, we realizes that, a quantity of developments which a priori have nothing to do with guesswork, in fact were motivated by that one ; a typical example is the whole theory of almost periodic functions of Bohr, the footballer, the brother of the physicist, so I mean, it's amazing, it's amazing. And about that, we explain it in detail in the book, what is important to know is that a mathematician faced with a problem, always has a technique which doesn't make him unarmed and what is this technique ? It is a very interesting technique which I think does not only apply to mathematics, it applies, I think, in fact to all kinds of fields ; and that's why I want to explain it.

It is a technique which consists in saying, faced with a fixed problem, for example Riemann's conjecture, instead of being there looking at the problem and then being unable to do anything, no. What we are going to do, the first thing we're going to do is criminal in some way. That is to say, we will take the problem and we will generalize it. So it sounds completely silly. It seems completely silly to replace a particular problem by a much more general problem. And the example that we take from the book, this is the example of chocolate bars. That is to say imagine we are there, we look at you, and then we ask you "what is the optimal way to break a 6×8 chocolate bar for example in small tiles ?". And then, the interest of generalizing is that we are going to hold power to specialize the problem generalized to much more simple cases. This is great because if you are faced with the problem of a 6×8 tablet, you're completely stuck because you say "but it's too complicated, I'll never get there." However, if you replace 6 and 8 with l and m, it seems weird. But now you take $l = 1, m = 3$, you have a shelf of three tiles, three tiles. Well, to break it, it's not very difficult. So in fact, in mathematics, we do that and we did that for Riemann's hypothesis, and it was something extremely fruitful because that is what allowed André Weil precisely, to demonstrate a generalization that had been made and to demonstrate that it was true in this case. So it gives confidence and in general, precisely, that makes it possible to give an anchor point, for what I was talking about earlier, that is to say the analogy. That is, once we have demonstrated a special case of the generalized problem, we have an extraordinary tool which is the analogy. That is to say that we imagine that the demonstration that we made in the particular case will be able to transplant, I do not say to transpose, I say to transplant, as I said earlier, with the small flowers which are very fragile. So she will be able to transplant herself in the case that really interests us. So the creative power of conjectures is not at all negligible, it is a kind of way of having seen further than the others, and after all, well, after, you have to get into the tough stuff, you have to try to demonstrate the conjecture.

Alain Prochiantz : But the conjectures are still proven ? Or are there some that are false ?

Alain Connes : Yes, of course, there are some that are false.

Alain Prochiantz : And if we work on a conjecture that is false, and if we draw interesting results, and we prove that it is false...

Alain Connes : In fact, what happens in mathematics is that there are two aspects. There is the aspect where there is a problem, is this problem resolved or not, well... This is an aspect of mathematics. But there is another aspect that is largely as important and it is the aspect of building theories. And for example Grothendieck was very well known for when he was asked a question, etc. he always tried to formulate the question in the good framework, and then he built a theory that made the question resolves by itself. Serre used the best metaphor in relation to that : he said that when we asked him a problem, he was trying to let it dissolve in a rising tide of general theories. So it says well what it means. And so in fact, there are the impetus which is given by a question such as a conjecture etc. Very often precisely, the most creative, the most positive aspect of a conjecture, it is the construction of the theories which will either resolve it or say it is false. It could very well happen. And besides, what we wants is to know the truth. We do not necessarily want to demonstrate. Actually, in the book, we tell a story that I don't want not to tell because the book, The Specter of Atacama, ends with this story ; this story is the story of an aging mathematician, well, think about whoever you want, who has been tackling a good guess for years, and who finally decides, because he sees that he is running out of time before him, to sell his soul to the devil, to know the answer. One says initially, to know the answer.

Alain Prochiantz : It's a known story, that.

Alain Connes : Uh, not so much the one we are telling...

Alain Prochiantz : The story of selling your soul to the Devil, in all case.

Alain Connes : Of course, of course. Selling your soul to the Devil is a known phenomenon, but in this case what happens is quite surprising because he ends up having a date with the Devil and moreover the Devil is embodied by Machine Learning. Huh ! *(laughs)*. So he ends up have a date with the devil and then when he meets the devil, they encounter in an ill-famed suburb of Naples and the devil begins by making him sign the papers that he sold his soul to the devil and the mathematician does not realize that because he signed the papers and that, he gave his soul to the Devil, he will change his behavior. So the Devil says to him "but hey what is your wish now ? You must give your wish..." And the mathematician says "I wish the hypothesis of Riemann's false" *(laughs)*. And it's only when he gets home he realizes that in fact what he just said is because he had sold his soul and that therefore, instead of wishing it to be true, etc., he wished it be false.

Alain Prochiantz : Dear Alain, I think we will close soon this interview which was really fascinating ; I would like to ask you a question : "Is mathematics a natural language for you ?"

Alain Connes : Well I think that not only it is a natural language, but I think it's the only language that will allow us to understand and to communicate with extraterrestrial intelligence. And it joins the book but I'm sorry to mention it too much, but well, what happens in the book precisely, it is that this message which is received and which is the spectrum of Atacama, it is received alternately with prime numbers, and an intelligence earthly, a mathematician, cannot fail to recognize an intelligence external to us, and which is manifested by this understanding which is extraordinary, which was made by Riemann in the XIXth century, so this is what I claim,... and there is a language that was invented called Lincos... , but what I claim is that we can communicate precisely with extraterrestrials thanks to mathematical language, why ? Because it is the only language that is not self-referential. It's the only language that is not self-referential, that is to say that, unlike a dictionary which when looking for the definition of a word, it refers to another word, which itself refers to another word etc. etc., the mathematical language is not self-referential.

Alain Prochiantz : But this language is composed, therefore, to return at the starting point, mental images.

Alain Connes : Uh no, this language is composed, at the start, by example of signals, which we send spectrally, which we send repetitively

Alain Prochiantz : But for example you, when you think ?...

Alain Connes : Ah when I think, of course, I think through mental images, of course.

Alain Prochiantz : You never think in natural language ?...

Alain Connes : No, no, no. Natural language, I mean, is a language which afterwards painfully tries to transcribe our mental images, our ways of thinking, etc., but I say "painfully" because in general, I can't get it to pass in a really satisfying way, I try to orally etc. there are people who are really strong at it and I think in particular of Grothendieck. Grothendieck was capable when, what we were talking about earlier, that is to say about an idea that was not still ripe, he was able to start writing on it and that,...

Alain Prochiantz : Did it ripen it ?...

Alain Connes : It made it mature, but I think it's not given to everyone to be able to write about an idea that is not ripe core and make it ripen, I mean.

Alain Prochiantz : To get it out of its cocoon...

Alain Connes : To take it out of its case, out of its cocoon. And there is anything else I wanted to say, before we finish, it is, I don't know if it was mentioned in another dialogue on the imagination, but there is an extraordinary example, it is the example of Eureka by Edgar Poe. So this example is still wonderful, to know that a poet has been able, in the XIX$^{\text{th}}$ century not only to have the intuition of the Big Bang, but to imagine the fact that the universe could then have a Big Crunch, etc., and that it could be mocked, he was mocked for over a century, until finally, we realize that in fact, he was right, but he was right by an intuition purely brilliant, and purely poetic.

Alain Prochiantz : Well, well, listen, I think it's the best way to end this interview with, I remind, you, Alain Connes, holder of the Analysis and Geometry Chair at the Collège de France. Thank you Alain, for coming today and see you soon.