A topo on topos
Alain Connes

Abstract : Alain Connes presents the intellectual approach that led Alexandre Grothendieck, from an "annoying" writing that he had to do for Bourbaki on homological algebra, to discover and perfect the concept of topos and he tries to explain in what sense this notion has a considerable scope, thanks in particular to the nuances it introduces between the true and false (Seminar organizer : Frédéric Jaëck (ENS), Transcription : Denise Vella-Chemla)

So I hope that I will stay in the spirit of the seminar and I think that the spirit of the seminar is Grothendieck, above all. So what I'm going to do is trying to get as far back as possible Grothendieck's thought and trying to explain precisely, as I said in my abstract, the journey that brought him to the topos, and above all, I will try to give you an illuminating metaphor for what is a topos, and explain to you what is extraordinary about this discovery, in the sense, especially for philosophers, in the sense that it introduces considerable nuances into the notion of truth. I will try to explain this by an example, because nothing like a good example to explain. There will be one of my slides called "at two steps from the truth" and I'm really going to give you an example of a topos which allows us to say that we are for example 10 steps from the truth, or that we are 15 steps from the truth, etc. So listen carefully. I'll make you hear Grothendieck, Grothendieck's voice, because Grothendieck did 100 hours of lectures in Buffalo in 1973, and in those 100 hours of lectures, there are things that interest us. Of course I will not make you listen to him for a long time but I will make you listen to a point in time, where he explains what a sheaf is to people who don't know this at all. And he explains how he will do his course on topos. We will see, there will also be an even more funny interlude at some point, we will not hear Grothendieck's voice but we will hear Yves Montand's voice. You will see, finally, you will hear all that.

So the first image I show you is an image that I owe to Charles Alunni who sent me an email one day telling me that he would have liked to have Grothendieck's second thesis. But at the time, when we were doing a thesis, when I did my thesis for example, there was always a second thesis. This second thesis was not written. It was a second thesis that we had to defend in front of the jury. And we had a subject that was given to us. What is quite extraordinary is that in Grothendieck's case, he did his thesis on nuclear spaces, on topological vector spaces and on nuclear spaces, and he made a fundamental contribution to functional analysis and what is extraordinary, it is that one can think that what made Grothendieck branch off, and which eventually led him to the idea of the topos, to this wonderful idea, this is his second thesis. Why ? Because the Grothendieck's second thesis is written in this text, it is on the sheaves theory.

And then on this page, if you are perceptive, you will find that there is an error, which shows that you are never safe from mistakes. Because there is one of the examiners, if you look closely, who is called *Georges* Choquet *(laughs)*. So I looked, to say maybe I was wrong, there are 3 examiners, there is Henri Cartan, there is Laurent Schwartz and then there is Georges Choquet. So I have searched on wikipedia to see if there was not a mathematician named Georges Choquet. In fact, no, there is a clergyman called Georges Choquet who died during the second world War. So there is no problem, it is a mistake, and it is Gustave Choquet who was Grothendieck's examiner. So he sustained his thesis in 53.

And already in 55, he was of course interested in sheaves, which was a wonderful discovery of Leray. And so there, I will start with some exchanges of letters between Serre and Grothendieck because finally, it is in these exchanges of letters that we see appear what is a so famous article that we call it The Tohoku. This article appeared in a newspaper called Tohoku Maths Newspaper but the article is so famous that in fact it is called Tohoku.

So that's what Grothendieck says ; he says :

"My Dear Serre,
Thank you for the various papers that you generously sent me, as well as for your letter. Nothing new for my part."

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So that justifies what I wrote when I announced my presentation :

"I finished my annoying writing of homological algebra."

So we will see very gradually what is the philosophy that Grothendieck uses all the time when he works, that is to say that he never hesitates before a task than any normal mathematician would consider it to be uninteresting, off-putting, not going to bring him anything. So he continues :

"... that I sent to Delsarthe who lacked the editorial staff for the typist."

He says :

"I proposed it to Tannaka for Tohoku ."

Tohoku is Tohoku Maths Journal.

"It seems that the river articles do not put them off."

So it is true that Grothendieck, in general, when he writes, the minimum is at least 100 pages. Then he talks about Weil. I'll spare you what he says because he says :

"I have read at least the statements in Weil's books on abelian varieties in the hope that we have arranged since the really discouraging demonstrations at Weil, his language disgusts me."...

More ! *(Laughs)* I pass... He says :

"I spend my time either learning or writing the varieties. It's fun but long sure, but there is no question of research until you have swallowed a mountain of new things."

So then, very important, this is probably the most important thing in this excha nge, he says to a given time :

"I realized that by formulating the theory of derived functors for more general categories than modules..."

(it must be said that at the time, there was the book by Cartan-Eilenberg which was in the making, Serre calls it the Cartan-Sammy - because it's Sammy Eilenberg - and in this book, there were of course the derived functors but it was still applied to module categories. That is to say, we took the category of modules on a not necessarily commutative ring and all the homological algebra was developed like that. But obviously, it was very analogous in its formulation with what was happening for cohomology with coefficients in a sheaf. So what Grothendieck says is :

"I realized that by formulating the theory of derived functors for more general categories as modules, we get the cohomology of spaces with coefficients in a sheaf at low cost."

You should know that at the time, when we took cohomology with coefficients in a sheaf, it was always Čech cohomology. That is to say that we took covers of the topological space, and then we made a complex or a bi-complex with these covers and we defined cohomology like that. Here.

So that's what he says, and that point in his correspondence is absolutely essential. So then he continues, and so this is a letter from June 4, 1955, so we are 2 years after his thesis, so :

"Attached is the result of my first cogitations in form, on the foundations of homology."

So afterwards, I won't detail the rest since it's on spectral sequences, etc. But let's say that there, Grothendieck explains that he had planted himself on the existence of enough projective sheaves but at that time, he demonstrated that there were enough injection sheaves, and that allowed him to define, if you want, cohomology with coefficients in a sheaf without assumption on topological space. If we have good hypotheses on topological space, at that time, it coincides with the cohomology of Čech. But this is not generally true. So here is the Serre's response. So there was something else in the correspondence, there was something else that was what is called Im_1, i.e. the functor for projective limits, how it commutes with the cohomology. But what is to read is the second paragraph.

"The fact that the cohomology of a sheaf is a special case of derived functors, at least in the para-compact case..."

(because in the para-compact case, it coincides with the cohomology of Čech, so the one which is defined from coverings)

"... is not in Cartan-Sammy."

(Cartan-Sammy, this is Cartan-Eilenberg)

"Cartan was aware of this."

So Cartan was aware that when they had developed the whole cohomological theory on the modules with Eilenberg, he was of course aware of the analogy with the case of the cohomology of sheaves, but hey, they didn't want to bother themselves to do it in their book, and Cartan had conscience and had told Buchsbaum to take care of it. So it's actually Buchsbaum who, independently of Grothendieck, also defined the abelian categories. He had started to develop it but he hadn't been involved in cohomology.

"But it doesn't seem to me that this one did. So the point of this would be to see what are just the properties of the fine sheaves to be used. So maybe we could realize if yes or no..."

(it's Serre speaking, of course)

"... there are enough fine sheaves in the non-separated case.".

So the non-separated case is extremely important of course for algebraic geometry and it was a time when Serre was developing algebraic geometry from Leray's sheaf theory taking Zariski topology and Zariski topology, at the start, it is not separate. I mean, so, we have exactly that problem. So this exchange is extremely important, it's an exchange dating from the year 55. And Grothendieck's article therefore, it is marked *Received 1 st March 1957*, so it is this article *"On some points of homological algebra"* which is really the ancestor, we can really place, if you want, the origin of the topos in this article.

The reason we can place the origin of topos in this article is that in this article, if you want, first, of course, he introduces what abelian categories are. So this is extremely important, with all their properties, etc. He develops homological algebra in the context of abelian categories. So that's what he does, if you want. But, what is much more important, is that he takes two types of examples, in his article, of abelian categories. The first example of abelian category that he takes, it is the abelian category of modules on a ring. It belongs to Cartan-Eilenberg. There is no problem there, but he also takes the example of the sheaves of abelian groups on a topological space, of course ; again, no surprise since it was for unify the two that he had done his generalization work. But what is absolutely crucial is that he had another example in mind, a third example in mind, and that's what he called the categories of diagrams. That is to say, what he was doing was that he had the idea that if you take diagrams of abelian groups, but whatever diagrams you look at, well, that still forms an abelian category. Well actually, if you think right, you realize that he had the two pillars of the concept of topos. That is to say, he had the notion of topological space, which gives the notion of sheaves on abelian groups, etc., and he also had the notion of categories of diagrams and we will see that these categories there, they give rise to a topos. And these topos have an absolutely fundamental role that we will use it all the time, all the time, all the time.

So there is one thing that should be noted : he defines what an abelian category is. So what Grothendieck says is that an abelian category is an additive category. So I will explain to you in two words the misconception that we have on the additive category which satisfies the two additional axioms following : then, there is the fact that a morphism must have a nucleus and a co-nucleus, that, it's an abstract notion, if you don't know it, well, it would take a while, it's a bit of gymnastics, and the second condition is being an exact morphism. That is to say, the fact that if you divide by the nucleus and if you look at what is called the image by defining it with respect to the co-nucleus, well then you have an isomorphism between the quotient by the kernel and image. And that is extremely important, it is not true for applications in topological spaces for example, if you take an Hilbert space and if you take the continuous linear applications in Hilbert space, that does not satisfy these two conditions. It is not an abelian category and the reason is that you can have a morphism which has an image, but it is dense in its image and its image is not closed and at this time, the second condition AB2 will not take place.

So there is a misconception that Grothendieck reproduces when he defines the additive categories in his article and that is the following : that in general, people define an additive category by saying "an additive category, it is a category where we add an additional structure which is the additive group structure on morphisms." Well, this is an heresy. I will explain to you why : because in fact it is not at all an additional structure. And there is an exercise in MacLane that I noticed for you there which shows that it is an heresy. What is this heresy ? The heresy is that when you take a category like an abelian category, it has products and co-products. This is a very simple thing. And it has an object which is both initial and final, it is object 0 ; in the abelian groups, the group is reduced to 0. Okay ? It is an initial object because you have a single arrow going from 0 to any abelian group, and it's a final object because you have a single arrow going from an abelian group to 0. Everything is sent to 0. So there is an object that is both initial and final. Well, if you have that, you can tell, when the category is abelian. How ?

Well, because what happens is that you have a completely canonical arrow, completely natural, which goes from the co-product of two objects to the product of two objects. Because using the 0, you can send half on 0 and the other half on 0 and at that time you have an arrow which is completely natural. If you asked if this arrow is an isomorphism, you did half the work. Because as explained in this MacLane text, at that time, you have an addition for morphisms. Simply by using the fact that this natural arrow which goes from the co-product towards the product is an isomorphism. You have a natural addition for morphisms and once you have this natural addition, you can take the additional axiom that there is a minus sign, if you want, for this addition, of course in general, you will not have a minus sign, that's what we call semi-additive categories, they are very interesting, but if you want a category to be additive, you just ask that there be a reverse. So then, it's not an additional structure. It is an heresy to believe that an additive category is given by a category + an additional structure. It's not true. So it's a misconception.

So, now I'm going to read Grothendieck, since the principle of the seminar is to fade away in front of Grothendieck. And even at some point, we will listen to him. And then when we have read and listened enough Grothendieck, there I will take a metaphor, then we will see an example. Okay so be patient, I'm not just going to read or listen to Grothendieck, you have to be patient, but let's listen to him anyway. Here is what he says :

"The point of view and the language of the sheaves introduced by Leray led us to look at these "spaces" and "varieties" of all kinds in a new light. They did not touch, however, the same notion of space..."

So what Grothendieck says is that Leray had envisioned a space in the form of sheaves on this space, but in fact, moreover, Leray only considered the sheaves of abelian groups. And we will see the change that Grothendieck has already made even there.

"They did not touch, however, on the very notion of space, only making us appreciate and more finely, with new eyes, these traditional "spaces", already familiar to all. But it turned out that this notion of space is inadequate to account for "topological invariants" essentials which express the "form" of "abstract" algebraic varieties (like those to which Weil's conjectures apply), even that of general "schemes"..."

So here is the moment when I will make you hear Grothendieck. Why, because we are going to hear Grothendieck who defines what a sheaf is. So, I think it's important that you listen to this, okay, because you don't necessarily know what a sheaf is, we're going to listen Grothendieck talking about this, okay, and once we've heard Grothendieck talking about it, we'll come back to our sheep. This is the start of his conferences in Buffalo.

"Topoi were objects for years essentially on general topology. I mean a topos could be considered as the main object of study of topology. And so topoi is generalization of classical general topology. It's what I really like to consider... Its study require some familiarity with handling topological spaces and continuous maps, and homomorphisms, and such things, and on the other hand familiarity with the language of categories... Later we... exponentiations... and give examples. But in order to understand... what a topos means..., one would require some familiarity with the language of sheaves on a topological space. I guess

that this notions are not very familiar for everybody, therefore I think I would give rather some introduction to sheaves on a topological space. I want to assume anything known on this matters. I would give a review of standard "sheafsury"... I hope that if some explanations are not clear, or if some comments come up that you'll freely interrupt me to ask questions, or to point out errors or to make any kind of comments. It would be nice if, as time goes on, there would be some kind of participation of the audience, I am sure that a number of you know something about topoi.... You will be able to make suggestions and comments.

So I start with a kind of formal survey of sheaves on a topological space. Provided an X to be a topological space, and consider the set $O(X)$, it depends on X, of all open subsets of X. The topology of X is defined in terms of the family of open subsets of X which is a subset of $PartiesdeX$. I recall the axioms of the topology is that O should be stable under arbitrary unions and under finite intersections. All right. So $O(X)$ in particular is an ordered set by inclusion and that, that all defines a category by abuse of language.

... any all upset does. If U and V are elements of $O(X)$, are open sets on X, the set of homomorphisms of U into V will be either empty if U is not contained in V and may be reduced to just the inclusion map of U into V, if U is contained in V ; this is the definition of empty and of the decomposition of arrows... There must be at most one arrow from an object to another. So this construction of a category in terms of open sets makes sense for any all upset whatever. So the category which behave as... arrows of the graph of all the relations... Now let's first define pre-sheaves : a pre-sheaf on X, say f, is by definition a functor which goes from the category $O(X)$ to the category of sets, when I say a pre-sheaf, I mean a pre-sheaf of sets, later we will see other types of pre-sheaves. But it should a contravariant functor... It's a functor which goes from the opposite category to $O(x)$ to the category of sets, so let's recall what this means for a functor : first of all, it means for the objects of that means for every open set, $U(X)$, we associate an object fU of the category of sets..., Remarque d'Alain Connes : ce qui est très important, c'est qu'il parle de faisceaux d'ensembles, et Leray parlait de faisceaux de groupes abéliens. *It is an arrow of categories, that means that for every inclusion map U to V two open sets, we associate a map from fV into fU..., this map will be called the restriction map, corresponding to the pref-sheaf, and the axioms are the evident axioms of transitivity, namely if we have V is contained in another open set W, then we have a restriction map from fV to fW but also from fW to fV and we want the restriction from fW to fU would be the composition here and moreover we want that in the case than we take the identity arrow, the identiry arrow from U into U, you want that the corresponding map fU goes to fU would be identity. ... as maps of functors... So a pre-sheaf on f is just a contravariant functor from the category $O(X)$ to sets is just O to sets. And the category of pre-sheaves on X, let us recall it $PreSheaf(X)$, is defined as being the category of all functors from $\dot{O}$ to sets. So the pre-sheaves can be viewed as being objects of a category, the category of pre-sheaves, in the category of functors.*

So what is an homomorphism of a pre-sheaf F into another G, by definition of homomorphisms of functors, let's say f be such an homomorphism by definition, f consists in a connection of maps from fU into gU, we call this map $f(U)$ for every U object of the category that of all open subsets of X, and these maps of sets will be compatible with restriction maps, that means that whenever U is contained in an open set V, then we have also fV that goes into gV by f of V and we have the restriction maps here, from fV to fU and from gV, to gU and so that the square should commute. So that's an homomophism of pre-sheaves, that is just an homomorphism of functors, and they compose in an evident way, their composing is out and ? ? ?. There is an homomorphism from the pre-sheaf... that means for every U, an homomorphism from gU into hU, the composant of g and f is defined as associating for every U the composition of homomorphisms here.

All right, so that's just general nonsense[1] *on functors, on categories. Now, so far, we have not used the fact that $O(X)$ was the category of open subsets of X, we have just used the fact that it's an ordered set, but we are going to use it now in order to define the notion of pre-sheaf on X which recall sheaves on X, we have to introduce another axiom on pre-sheaves which will turn on to sheaves. Now, traditionnally the axioms on pre-sheaves is separate in two : you say first that the pre-sheaves are separators if the following is true : for every open set U on X and for every covering of U, open subsets U_i a covering is a union of the $U_i s$, can be defined in terms of ordered sets ; it's just the supremum of the $U_i s$, in terms of ... Be f a mapping of fU into each one of the $fU_i s$, a restriction map, and therefore, we get a mapping of f into the product of the $fU_i s$ and have that two sets separated by every such choice here, this mapping is injective.*

Now, let's state this in another way : if f is a pre-sheaf on X, f of U, the elements of the sets accord

1. ?

to sections of f over U. When we'll give examples, we will see where this accord of sections come from. All right, and, the map here I already said contained in V, the given map of fV into fU will be called the restriction map… that mean the sections of … is called the restriction of … to U and the axiom for a sheaf to be separate means that whenever there is a union of open subsets $U_i(X)$, from which the union is U, there's a section of the pre-sheaves over U is known as we know these restrictions on the U_is. The first condition would be the restricted in … but in geometric terms, it just means that … is an injective arrow, that means that a section of f over U can be identified for the system of sections of f over these U_is, and then the second question that arise is to see whether we can identify the subsets here that we obtain as image of fU which are the systems of sections of f over U_is which come from global sections f over U. Now let's take a system of sections say Φ_i in fU_i for every i, for every index, here the necessary condition for this system of Phi_is to come from global sections which is the following : when you have two indices i and j, the restriction of Phi_i to $U_i \cap U_j$ could be equal to Phi_j…"

Here, I stop here, our patience is probably exhausted. But, I will say that one of the reasons for which I made you hear Grothendieck is quite complicated : we have to get used to this incredible patience he has to explain all the details, to come in, to go through all the details. And that, we will see, I mean, it is an absolutely essential quality in his approach. So I continue to read what he said about the point of view and the language of sheaves introduced by Leray. Grothendieck continues, he says :

"*For the expected "marriage of the number and the size", it was like a bed definitely narrow, where only one of the future spouses (that is, the bride) could at least find a place to nest somehow, but never both !"*

So there, if you will, he actually had in mind all kinds of developments that were of combinatorial nature, which were related to number theory as opposed to what was going on in topology but hey, of course, there was Serre's work on the use of Zariski's topology.

"It is the point of view of the sheaves which has been the silent and sure guide, the effective key (and by no means secret), leading me without procrastination or detours towards the nuptial chamber with the vast marital bed. A bed so vast indeed (like a vast and peaceful very deep river…), that
"All the king's horses
could drink together…""

We'll come back to that.

"As an old tune tells us"

(which I will make you hear later)

"As an old tune tells us that you must have been singing too, or at least hearing sung. And whoever was the first to sing it felt better the secret beauty and the peaceful force of the topos, that none of my learned students and friends of yesteryear…" (laughs)

Well, there is of course in this sentence, the fact that we know, without doubt, many of us, which is that the first to reveal a certain mathematical landscape has an apprehension of it which is incomparable (this is what happened to Galois for example) compared to the other mathematicians who come after him and understand him. This is a very striking thing, he says something more nasty, of much nastier.

"The key was the same, both in the initial and provisional approach (via the very convenient concept, but not intrinsic of the "site")" (which I will tell you about, of course) *"… than in that of the topos. This is the idea of the topos that I would now like to try to describe.".*

So let's let Grothendieck speak for sure.

"Let us consider the set formed by all the sheaves on a (topological) space"

So, for the mathematician, frankly, these are sheaves of sets, that is absolutely fundamental, it is an enormous step, that it replaced the sheaves of abelian groups, which were interesting, we thought they were the only interesting ones, since they are the only ones who will give a cohomology, etc. No ! He had the idea, which seems completely naive, to replace the sheaves of abelian groups by sheaves of sets, and we will see the range it gives.

"I believe elsewhere" (and that's what he says) *"to be the first to have worked systematically with sheaves from 1955.*

What is the advantage of working with sheaves of sets, as we will see, is that when you work worth with the sheaves of sets, you can define what a group is in this stuff, you can define what an algebra is in this thing, because what you do is that you work as if you were working in sets but there is variability. That is to say, there is something that moves, but otherwise you do exactly as if you were working in sets, as usual. So you can define what a group is. So if you're looking for what an abelian group is in this category of sheaves of sets, well, you find the sheaves of abelian groups. So we fall back on our feet.

Then what Grothendieck says :

"We consider this "set" or "arsenal" as having its most obvious structure, the one that appears there, if one can say, "at first sight"; namely, a so-called "category" structure."

So of course, we were educated, at least in my time, with set theory. In fact, it was probably a mistake. The real way of thinking is category theory. So he thinks of this kind of theory that he has before his eyes as a category.

"(Let the non-mathematician reader not be confused, not to know the technical meaning of this term. He won't need it for the future.)"

That is to say, what the non-mathematician reader simply has to think is that he has an analogous, now, of set theory, in this new category, in this category of sheaves of sets. There is something that resembles set theory, and we will see that this metaphor goes very far.

"It's this kind of "surveying superstructure", called "sheaf category" (in space envisaged), which will henceforth be considered as "embodying" what is most essential to space."

So we will see a metaphor, which I will develop further, but I can already reveal a part of it. If you want, usually, when we talk about a space, I'll show it to you right away because I don't want to wait, for this metaphor. Here, the metaphor is as follows : if you want, before Grothendieck, we used to, when we were studying a space, I don't know, a curve, or wearing what, we put space... on the stage. And then we looked at it, we studied it, as a whole with structure, etc. Well, what Grothendieck does is... no! Space is not on the stage, space, it's behind the scenes. On the scene, there are the usual actors of set theory : the abelian groups, algebras, etc. But, the space in question, it is behind the scenes, like a species of Deus ex machina which introduces variability in the characters who are on the scene. That is to say that now the characters who are on the scene, they will depend on a hazard. This hazard is governed by the topos. And when we have a Grothendieck topos, there are also the constants, that is to say that there are also the sets which do not depend on the hazard. And cohomology, it is defined by comparing the two.

So it's absolutely fundamental that you gradually try to acquire a mental image, even if you are not a mathematician, to understand that the space which is given by the topos, it will appear behind, it is behind the scenes, it is not at the front of the stage, it is not it that we studied; we study set theory, but there is this good god of topos, which is hidden and which makes everything vary, which introduces variability in there, okay. So then, that's what Grothendieck says in another way : what he considers as *"this set or "arsenal" as having its structure the more obvious"*, he thinks of it as a category, this category of sheaves of sets. So then what's he saying? *"That it is a lawful thing to forget the space and to consider only the category of sheaves of sets."*

Why is it a legal thing? It's a legal thing because we haven't lost space along the way. We find the points of space. So how do we find the space points in the metaphor I gave you? Because he says to him *"it's a simple exercise to check : once the question is asked."* . Well. You can actually have fun, thinking in classical terms, if you have the sheaves of sets on an ordinary topological space, how are you going to find the space itself, that is to say the points of space? Can you ask this question. So in fact, in the metaphor I gave you, the points of space are when you take a given moment, a fixed time. When you take a time that is frozen, well then there is no more variability, and you have ordinary set theory. That's what we abstractly call in the theory of topos a point of a topos, that is to say that this is how we call a geometric morphism, which goes from ordinary set theory to the topos considered. But that is exactly like freezing things at a given time.

Grothendieck tells him *"to check is a simple exercise."*. In fact, this is not always true ; as he says *"(at intention of the mathematician) Strictly speaking this is only true for so-called "sober" spaces"*. There still must be a minimum of separation in space. And there is an extremely interesting, which I invite you to do, because you always have to... we don't do math by listening, we do math by doing exercises. So there was already the exercise on the abelian categories of everything on time. And that's another exercise now. Is that you take a curve with its Zariski topology. Or you rather take a surface with its Zariski topology. And you calculate the points of the corresponding topos, i.e. of the topology of the sheaves for the topology of Zariski. Well, you will notice that there are more points, because the space in question, it is not sober, and you will get exactly the points of the corresponding diagram. So I'm going to say, already, in this example, we see the absolutely incredible potential of this way of thinking. He says :

"... we can now "forget" the initial space, to no longer retain and use only the associated "category", which will be considered the most adequate embodiment of the logical "topos structure" (or "spatial") that needs to be expressed."

So what he explains next is :

"As so often in mathematics, we have succeeded here (thanks to the crucial idea of "sheaf", or "cohomological meter") to express a certain notion, (that of "space"), in terms of another (that of "category")."

So we replaced, always following the metaphor, space by this category, which is a category type of sets, these are sets.

"Each time, the discovery of such a translation of a notion (expressing a certain type of situations) in terms of another (corresponding to another type of situation), enriches our understanding."

Of course.

"And both, by the unexpected confluence of specific intuitions that relate either one or the other. Thus, a situation of a "topological" nature (embodied by a given space) is here translated by a situation of an "algebraic" nature (embodied by a "category") ; or, if you like, the "continuous" embodied by space, is "translated" or "expressed" by the category structure, of an "algebraic" nature (and hitherto perceived as being essentially "discontinuous" or "discrete")."

We deny, a priori, to a category the right to represent something continuous. This is the case here. This is the case because we find the points and we find the topology of the points, simply from the category, which *looks like* the category of sets.

"But here there is more. The first of these notions, that of space, appeared to us as a notion in a "maximum" way - a notion so general already, that one can hardly imagine how to find for it one more extension that remains "reasonable". On the other hand, it turns out that on the other side of the mirror, these "categories" (thus the categories which one obtains as categories of sheaves of sets) *"that which one finds, starting from topological spaces, are of a very particular nature..*

The "mirror" in question here, as in Alice in Wonderland, is the one that gives as an "image" of a space, placed in front of it, the associated "category"."

So this category if you want, it is the category of the scene behind which is the topos.

"They indeed have a set of strongly typed properties, (we will not talk about them all immediately) *that make them look like sort of "pastiches" of the simplest imaginable of them."* What is the simplest of them ? It is set theory. So the categories you get like that, from a topos, are pastiches of set theory. That's what Grothendieck says.

"That said, a "new style space" (or topos), generalizing traditional topological spaces, will be described simply as a "category" which, without necessarily coming from an ordinary space, has nevertheless all these good properties." So they're called topos, so he's going to say it by the way, wait, I have to find him...

"The name "topos" was chosen (in combination with that of "topology", or "topological") to suggest that it is the "object par excellence" to which topological intuition applies. Through the rich cloud of images mentally aroused by this name, it must be considered to be more or less the equivalent of the term (topological) "space", with simply a greater emphasis on the "topological" specificity of the concept. So..." Well, he's talking about vector spaces, etc. So I go back.

*"So here's the new idea. Its appearance can be seen as a consequence of this observation, almost childish to tell the truth, that what really matters in a topological space is nully its "points" or its subsets of points, and proximity relationships, etc. between these, but that these are the sheaves on this space, and the category they form. All I did was to lead towards its ultimate consequence the Leray's initial idea - and this done, **take the plunge**. Like the very idea of sheaves (due to Leray), or that of the diagrams, like any "big idea" which shakes up an inveterate vision of things, that of topos has what to disconcert by its character of naturalness, of "evidence", by its simplicity."*

In fact, we know when we do math, when we are on the right track, when someone tells you *"Oh, that's it !". (laughs)*

This is what Grothendieck says, so :

"... by this particular quality that makes us shout so often : "Oh, that's it !", in an half-disappointed, half-envious tone ; with in addition, perhaps, this implied "wacky", "not serious", that we reserve often to all that confuses by an excess of unforeseen simplicity. To what comes to remind us, perhaps, the long buried and denied days of our childhood..."

So there, he comes back to the concept of schema :

"It constitutes a vast extension of the concept of "algebraic variety", and as such, it has renewed from top to bottom, the algebraic geometry bequeathed by my predecessors. That concept of topos constitutes an unsuspected extension, to put it better, a metamorphosis of the concept of space.".

If you will, which is absolutely extraordinary, even at the outset, in the concept of topos, it is the way space is understood. As I said, it is no longer apprehended by points, it is apprehended by the hazard it introduces : it introduces a hazard into set theory ; he introduces a variability in set theory. And that is extraordinary.

"By this, it carries the promise of a similar renewal of the topology, and beyond this, of geometry. From now on, moreover, it has played a crucial role in the development of new geometry."

This is for l-adic cohomology, or for crystalline cohomology.

"Like her older sister (and quasi-twin) [2]*, she has the two essential complementary characters for any fertile generalization, here it is."*

First, this notion should not be too broad, I pass quickly enough on it ; he talks about topos, we talked about it ; *"the most essential geometric constructions must of course apply, that they can be transposed in a more or less obvious way."*. It shouldn't be too general, the notion you take should not be, for example, the general notion of category. If it was the general notion of category, we would not go far. So it must have this property.

He explains : *"Among these "constructions" , there is in particular that of all familiar "topological invariants"."*

He explains very well : *"For the latter, I had done everything necessary in the article already quoted from "Tohoku" (1955)."*

2. schema theory

So the origin, you see, it comes from there. As I said earlier, in Tohoku's article, there were both the sheaves on topological space and there was also, and it was extremely important - so much important, the categories of diagrams, he was talking about the categories of diagrams and we will see that they play an absolutely essential role, in the same way. So he talks about mental associations, and less technical notions, of course.

"Second, the new concept is at the same time broad enough to encompass a host of situations which, hitherto, were not considered to give rise to intuitions of a "topologico-geometric" nature - to the intuitions, precisely, that we had reserved in the past only for ordinary topological spaces."

As we will see in an example that I will give you in a relatively short time, as we will see, what happens in the metaphor I was talking about, what will happen is that as it there is this hazard, as there is this variability in set theory, we can no longer apply the excluded middle principle. On the other hand, intuitionism works. And so, what it will generate, this nuance, that's why I want to take you there step by step, we're slow, but we have to be slow, so I will show you an example, as I said at the beginning, in which the notion of truth associated with topos will be much more subtle than the notion of ordinary truth, and I will have a slide on which there will be marked "a step from the truth" and I will give you a topos in which we will be "10 steps from the truth", "15 steps from the truth", "20 steps from the truth", etc. We will take an example because until we have taken an example, as long as we talk abstractly, we don't really know what we're doing. So we're headed there.

"The crucial thing here, from the perspective of Weil's conjectures, is that the new notion is pretty vast indeed, to allow us to associate with any "diagram" such a "generalized space" or "topos" (called "étale topos" to the proposed scheme). Certain "cohomological invariants" of this topos (all that there are "dumb kids"!) seemed to have a good chance of providing "what we needed"".

He continues and we relax a bit before coming to the examples and the really crucial things.

"It is in these pages that I am writing that, for the first time in my mathematician life, I take the leisure to evoke (if only to myself) all the master-themes and great guiding ideas in my mathematical work. This brings me to better appreciate the place and the scope of each of these themes, and of the "points of view" they embody, in the great geometric vision that unites them and from which they come. It is through this work that the two innovative nerve center ideas appeared in full light, in the first and powerful development of new geometry : the idea of diagrams, and that of the topos."

And there he insists :

"It is the second of these ideas, that of topos, which now seems to me to be the deepest of them. If by chance, towards the end of the fifties..."

So Grothendieck introduced the topos in a slightly depressed period he had after the death of his mother in 1957, he introduced the topos in 58. So we will be, in the coming year, in the 60th anniversary of the birth of the topos.

"If by adventure, towards the end of the fifties, I had not rolled up my sleeves, to develop stubbornly day after day,"

This is Grothendieck : "Obstinately, day after day..."

"Throughout twelve long years, a "schematic tool" of delicacy and power perfect - it would seem almost unthinkable to me that in the ten or twenty years already that have followed, others that I in the long run could have prevented from introducing at the end of the ends (were it to their bodies defending) the notion which obviously imposed itself, and to draw up somehow at least a few dilapidated "prefab" barracks, failing the spacious and comfortable[3] *residences that I had at heart to assemble stone by stone and go up with my hands."*

3. address of the speaker to the audience : "there, he is talking about the diagrams"

There he talks about schemas.

"On the other hand, I don't see anyone else on the mathematical scene, over the past three decades, who could have had this naivety, or this innocence, to take (in my place) this other crucial step between all of them, introducing the so childish idea of topos (or even that of "sites"). And even assuming that idea already graciously provided, and with it the timid promise it seemed to harbor..."

You know, someone would tell you, "I'm going to do this...". "Good luck !", you would say ! Okay... *(laughs*[4]*)*

"... I don't see anyone else, either among my friends of yesteryear or among my students, who would have the breath, and above all faith, to bring this humble idea to fruition (apparently ridiculous...) "What is it to strive on the sheaves of sets on a topological space ?"... (seemingly derisory when the goal seemed infinitely distant...) : since its earliest early beginnings, until the full maturity of the "mastery of eternal cohomology", in which it ended by incarnating in my hands, in the years that followed."

Good, after, he talks about details, finally, of things which are important for the mathematician, but he speaks of etale cohomology and it is on this subject, as he says :

"It was inspired by this statement that I discovered the concept of a site in 1958." So he discovered the notion of site in 1958, and it is this notion, of course, and cohomological formalism, which were developed later.

He says :

"When I speak of "breath" and "faith" ," (that's always for the mathematician), *"these are qualities of "non-technical" nature."*

Grothendieck wrote somewhere in *Crops and Sowing* that he was not fast, that he was surrounded by people much faster than him, but hey, this is an example that shows how not to be discouraged, when you are not fast, good when talking with people who you realize they understand ten times faster than you do, don't be discouraged. However, which is absolutely crucial is to be persistent, and to have faith in an idea.

That is to say, if you have an idea, you must first make it your own, make it your own. And once it's yours, you have to protect it ; initially, you have to protect it like a very small child who has just been born. Do not show it too much, not too much, etc. *(little laughs)*. And then after... Not so much because someone can take it from you but because you have to test it, we will talk more about it late, you have to test it, you have to get used to it, in private.

"On another level, I could also add to it what I would call "cohomological flair", that is to say the kind of flair that had developed in me for building cohomological theories."

Afterwards, he groans a little against his pupils, but that, we are used to with Grothendieck.

We're going to take a short break : we're not going to stop but I'm going to make you listen to Yves Montand *(laughs)*.

"Yes, the river is deep, and vast and peaceful are the waters of my childhood, in a kingdom that I thought I quit a long time ago. All the king's horses could drink there together at ease and all their drunk, without exhausting them ! They come from glaciers, fiery like these distant snows, and they have the softness of the plains clay. I just talked about one of these horses, that a child had brought to drink and who drank his content, at length. And I saw another one coming to drink a moment, in the footsteps of the same kid if it is - but there it did not drag. Someone must have chased it away. And that's all, as much to say."

We hear to the song Aux marches du Palais who speaks about The girl who has so many lovers that she doesn't

4. to underline the irony of the euphemism of Good Luck compared to the magnitude of the task that this represents.

know which one to take and who has chosen her little shoemaker.

Here we go back to serious things.

So he goes on, he says :

"However, I see countless herds of thirsty horses roaming the plains - and not later that this morning their whinnies pulled me out of bed at an undue hour, I am going to my sixties and I love tranquility. There was nothing to do, I had to get up. It makes me pain to see them, in the state of lanky rosses, while good water however is not lacking, nor green pastures. But it looks like a malicious spell has been cast on this land that I had known welcoming, and condemned access to these generous waters. Or maybe it is a coup mounted by the country horse traders, to bring prices down, who knows ? Or maybe it's a country where there are no more children to lead the horses to drink, and where the horses are thirsty, for lack of a kid who finds the path that leads at the river..."

So, with Pierre Cartier and Olivia Caramello, we organized a conference two years ago, in the IHES, precisely to revive the idea of topos, but really of Grothendieck's topos, and the colloquium was remarkable, it went very very well.

So what is a Grothendieck topos ? So now we come back to mathematics. So there are three ways of defining them : it is undoubtedly the first way which is the simplest. So, if you will, as Grothendieck explains when he lectures, the important thing, at the start, they were pre-sheaves, that is to say they were contravariant functors which went from the open category, but it's an extremely simple category. I mean it's a category for which between two objects, there is at most a morphism so it's really something extremely simple. So we looked at the contravariant functors that went from this category to the sets. So now, we remove all the conditions on this category, except that it is a small category. What does a small category mean ? It means that the objects form a whole. And then morphisms as well of course. So we look at a small category and we look at all the contravariant functors from this small category to sets. We forget the fact that we had them open and that we had an extremely special category by looking at the open. Okay, we take no wear which. And so now, what we're asking, of course, we're not going to take all the contravariant functors, since we know well, and Grothendieck explained it in what we listened to, that we are not going to take all the pre-sheaves. Among these pre-sheaves, we will select some which we will call sheaves. What is the important property of this selection ? There are two things that are very important in this selection : the first is that we are not going to change the morphisms ; the first thing that is fundamental is that when you take a morphism from one sheaf to another, you can forget that these are sheaves. It's a pre-sheaf morphism, okay. So in fact, when we are going to select the subcategory of the pre-sheaf category, we will take a full subcategory. Full subcategory, that means that we are not going to change the notion of morphism. Okay, that's crucial, that, if you listen to Grothendieck further, he talks about it and he says it is crucial. So first thing. Second thing, which is extremely important, is that there is a way, when you have a pre-sheaf, to harness it, to turn it into a sheaf. So that means that sheaves are special pre-sheaves, but there is a kind of projection that allows you to replace a pre-sheaf by a sheaf. So what is the correct way to say it is that the functor which includes the category of sheaves in the category of pre-sheaves, first it is full, it is faithful, because we forget nothing, but above all, it has an adjoint on the left, who is the harnessing, and miraculously, this adjoint on the left, it is exact on the left, which is normally never the case for an adjoint on the left. Normally, an adjoint on the left, we know that in all cases, it will preserve those which are called colimits, but it is very rare that it preserves the limits. Well there, it preserves the limits. So that's the condition. So if you want a short definition of what a topos is, that's it.

So now what we will see, and then we will see what a site is, there is another way to say it which is more precise : it is that in fact, we know that any harnessing, like the one of which I was speaking, in fact, it comes from what is called a Grothendieck topology on the small category which we left. So let's see what it is. And then in fact, there is a third definition of what is exactly a topos. But then, that is really a definition, how to say, very abstract but it is a definition that states properties that are true for set theory, okay. And we ask the topos to check them too. So that spawned another theory of topos, called the elementary topos theory, which are not Grothendieck toposes in general. But then, what is it missing to an elementary topos, therefore a topos which verifies naive properties of set theory, to be a Grothendieck topos ? What it lacks are the constants. That is to say that when, in the metaphor, I told you that we have variable sets, which depend on a hazard, well, when we have a Grothendieck topos, there is what is called a geometric morphism, which goes from the topos to the topos of the sets, and that allows us to talk about the constants. Now to talk about the constants, when doing cohomology for example, it's absolutely essential. Because these are the constants which allow for example to define the global sections of a beam, etc., etc. So it's not at all innocent, and there is a very big difference between a Grothendieck topos and what is called an elementary topos which would bring together elementary properties of set theory.

So the examples. Well then among the examples, there is of course the example of sheaves of sets on a topological space. This is the first example. The second example, I already talked about it, are sheaves for Zariski topology, so this is a special case of sheaves of sets on a topological space. But as I said, the point is that when we look for the points, for this topos, we find the good points of the diagram, okay. And finally, there is a third example, which is the one that Grothendieck introduced in 58 to have etale cohomology, that is to say that we start from a diagram, and there is a topos which is associated with the diagram, but it is no longer a topos which comes from a topology on the diagram. So this is something that is above and which, well, of course, there is already a topos in the sense original if you want, which is outside the topological spaces.

So I come back to Grothendieck, he says :

"The theme of the topos comes from that of the diagrams, the same year in which the diagrams appeared - but in extent it goes far beyond the mother theme. It is the theme of the topos, and not that of the diagrams, which is that "bed", or this "deep river", where geometry and algebra, topology and arithmetic come together, mathematical logic and category theory, the world of the continuous and that of structures "discontinuous" or "discrete". If the theme of diagrams is like the heart of new geometry, the theme of the topos is the envelope, or the abode. It's what I have designed more broadly, to grasp with finesse, through the same language rich in geometric resonances, a common "essence" to situations of the most distant from each other, coming from such and such a region of the vast universe of mathematical things. This theme of the topos is very far from having known the fortune of that of schemas.".

There is a kind of curse on topos. There is a curse that reigns, we may come back to it if we have time. So here is the metaphor. So the metaphor I was talking about earlier. It is absolutely necessary that you have a mental picture of what a topos is. So we used to, like I said, to put the space to be studied on the front of the stage. Grothendieck makes it play this role of Deus ex machina, who is not present, who stays behind the scenes. But what is important is to know that when you have a topos, you can do all the manipulations, you can talk about abelian groups, you can talk about algebras, etc., and if you work with a topos coming from a topological space, that would give you the sheaves of abelian groups, or the sheaves of algebras, etc., it's great, it's great to have this freedom of maneuver. So then, when one works in a topos, everything happens as if we were handling ordinary sets. So that's what one needs to know. In fact, as soon as we have bundles on a space, we get into the habit of thinking about a bundle as in a variable vector space. But there, variability is the **right** notion of variability, because it sets up the sets. Except that we can no longer apply the rule of the excluded middle. So what appears if you want is that we can no longer have, for a proposition p, (p is true) or (not p is true), we no longer have the excluded middle principle. So we will quickly see a concrete example of a topos for which this notion of truth becomes more subtle than the simple true or false than we use colloquially. For example, if you want, if you watch TV, and you watch a political discussion on television ; well, we used to say "this one is right and this one wrongly". Well, I pretend that we don't have the conceptual tool we need to judge. And I will give you examples. I'm going to show you how much more subtle the notion of truth is and how the idea of the topos allows to formalize it. So we're going to make this work on an example. To make that work, we are going to introduce topos which are other than the topos which come from a topological space and which have an extremely simple nature : these are the topos which consist in taking a small category and simply take the category of all contravariant functors to the sets. So here we does not distinguish between sheaves and pre-sheaves. We take all the pre-sheaves. They are all sheaves. So to a small category, we are going to associate a topos which is its kind of dual, if you want, which is all the contravariant functors from this little category to the sets, and we will have fun with that.

So in 91, Grothendieck was still in perfect contact with certain mathematicians, and lo and behold what he wrote in a letter to Thomasson, he said :

"On the other hand for me, the original paradise for topological algebra is by no means the simplicial category..." . So, I don't know if we will have time to talk about it, but he is talking about topos.

Indeed, topos having categories sheaves on sets $\hat{C}$, with C a small category, are by far the simplest of the known topos." And it is for having felt, that he insists so much on these categorical toposes in SGA4. So if you look at SGA4, you will see that there are two basic topos examples ; of course, there is the etale topos, and then there are the topos which are dual of a category. Okay. So we're going to have fun with that.

So what is the notion of truth in a topos ? *(laughs)* How is the notion of truth different in a topos ? So, in what sense, first of all, are we able, in the sets, to define the true and false ? So how are we going to define true and false in set theory ? We are going to be interested in trying to classify the subsets of a set, okay. You see, if you work with ordinary sets, and if I tell you "I have a functor which, if you give me a set, it combines all of its subsets." . It is a functor because if you have an application that goes from X in Y, you can recall the subsets of Y backwards, so it's a functor. So now, the question is "is this functor representable ?". It's a mathematical notion, okay, and then in the sets, it is representable because of a notion that we know very very well : it is that with a subset, we associate what is called its characteristic function. That is to say that when we have a subset of a set, we define a function : this function, it is 1 if we are in the subset, and it is 0 if we are not in the subset. So this function, this makes that it has a fairly miraculous property : it classifies, that is to say, it represents this functor. In the case of sets, there is a privileged object Ω which is the object which is formed of the set $\{0, 1\}$, the two-point set, and when you look at all the subsets of a set, it comes back to look at all the applications of this set towards the set $\{0, 1\}$. Because when you have a application which goes to the set $\{0, 1\}$, the subset, it is defined by the subset on which it takes the value 1. But where it does not take the value 1, well, it necessarily takes the value 0. Well, if we think enough, in logic, thinking in the language of topos, we realize that it is this simple fact that there were only 0 and 1 in the sets which makes it possible to have the principle of the excluded middle.

So now we're going to have fun with a topos which is a little bit more complicated. We will take... this is what I call "at two steps from the truth". So, what are we going to take as a topos ? We will take the category C which has only one object, and which has for morphism the powers of one morphism. That is to say that I choose a single morphism which goes from this object in itself and I raise it to powers, okay, T^n. So what does that mean, an object from the category of contravariant functors from this category to sets ? It simply means a set with an application from X in X. That's all. Why do you only have one set ? Because the category had only one object. So you only have one set. And in fact, the category, it had only one morphism, well, we raise it to its powers, but, I mean... you just have to know it, you just have to know its image. What is its image ? It is a transformation T from X into X. What is the topos ? Well, these are the sets provided with a transformation. Well, the sets with a transformation, that makes a topos. It's a category, okay. Why is it a category ? It's a category because if you have two sets with a transformation, you have the applications from X to Y which respect transformation. That is, they verify that the image $f(TX)$ of TX is $Tf(x)$. So you have a category, and this category is a topos. Why is it a topos ? Because it is the dual of the small category that I gave you.

Okay so now we're going to look for Ω, for that, so we will try to classify the sub-objects of an object. So why is it annoying to try to classify the sub-objects of an object ? Well, let's try with $\Omega = \{0, 1\}$. We will try with the characteristic function, as we did earlier. After all, if I take a set with an application, if I take a sub-object, it's a subset that is stable by application, okay. So if I take my X, I will take a subset Y which was invariant, which was invariant by applying T. Well. Very good. It is invariant by applying T. So I'm going to associate the value 1 on this subset, okay. On the subset, I will give 1. Why can't I give the value 0, on the complementary ? Well because there can have complementary points that will end up landing in the set in question. I'm not at all assured that the complementary will be invariant under T. It may very well happen that a point in the complementary, after a while, tac !, it will type in the subset in question. It's not because the subset in question is invariant that its complementary is invariant. Of course no ! The transformation I took is an action on N, not on Z. So how do we go do ? It's annoying ! It means that the application which went towards 0 and 1 does not work. Well ! Well, you have to think about it a little bit. What do we have to do ? Well, when I take an x which is in X, there will exist a smaller integer, a first integer, such as when I apply T n times, it hits the subset. Okay. So I'm going to associate this integer with it. This integer, it will be infinite, of course, if we never get into the subset.

So we see that we have to replace the set $\{0, 1\}$ by the set $\{0, 1, 2, 3, \ldots, \infty\}$. And how are making this set a set with a transformation ? Well, we realize that if I look the h, i.e. the smallest integer for TX, well the smallest integer for TX, it's going to be the smallest integer for X minus 1 unless it becomes negative ; if it becomes negative, it doesn't work ; so I take the *sup* with 0. Okay.

So you see that for this topos, then the notion of truth which before was simply 0 or 1, main-holding, it is given by the set $\{0, 1, 2, 3, \ldots, \infty\}$, with the transformation which replaces by $N - 1$. So what does that mean ? Well, that means we have an incredibly simple example of a topos which allows to say "Yeah, what you do, yeah, I would say it's 10 steps from the truth...". Me, I've always said that people who do string theory are infinitely close to the truth. *(laughs)*

So you see that this notion, innocent as it is, stupid as it looks, in fact, it has an absolutely rich potential. And what I claim is that our mind, our training logic, is extremely primitive because we are used to, when we listen to a discussion policy of decreeing "yes or no", "such a person is right, such a person is wrong" and we are wrong by doing that and if there were philosophers, well, I dream, if there were philosophers knowing math, and who understand the topos from the inside, and there are very few, who understand the topos from inside, they would be able to give models, which would be useful, to appreciate much better these kinds of discussions, these kinds of situations, which are actually much more subtle compared to the notion of truth, that this notion of absolute ineffectiveness, that we use all the time, and that is "Such is right or so is wrong.". So I absolutely wanted to give you this example, so that you keep it in mind, and try to build other similar examples ; there are examples of course, don't be afraid that there are n that goes from 0 to infinity, in fact, you can very well imagine finite constructions, okay. The finite constructions, there is a wealth combinatorial in the topos which makes finite constructions have extraordinary potential.

So what is a sieve ? This example will allow us to define what a sieve is. What is a sieve ? I gave you an example of a sieve. The Ω in general, when you take the topos, which is given by all the contravariant functors from a small category to sets, well we build the Ω and how is the Ω built ? The Ω, it is constructed from a sieve. So what is a sieve ? Well a sieve on an object of a category, the Ω will be constructed from the objects of this category, let's remember that the category which I mentioned earlier, it had only one object. So for the moment, we have nothing. We have a single object. So, a sieve on an object X, on our object X, is the data of a family of morphisms $C(X)$ which is contained in all the morphisms whose image is X... finally, it goes from a set Z to X, whose codomain is X, and which is stable by right-hand composition.

What are cribles, in the earlier example ? We had only one object ; the morphisms which went into this object, it was just the integers, since it was the powers of T, there was T^0, T^1, T^2, ... What is a sieve ? Well, a sieve is a kind of ideal if you will, that is to say that it is a family of morphisms which is stable by right-hand composition by any what morphism. So in the case of earlier, what is the composition on the right ? It add to an integer, well, that adds any integer to it. It's like looking at all the intervals infinite on one side. So, among the infinite intervals, you have what, you have 0, up to infinity, that is what is called... finally, it is a sieve which must always be present ; it is the sieve that is formed by all morphisms. And then, we had all the morphisms that were from a certain integer n. That was a sieve, okay, and it was when we were at a distance n. And then, there is the sieve where there is none to none, it's the empty set, and that corresponded to infinity earlier. Here.

So it turns out that in general, we can define the Ω, the truth values if you want, for the dual of a small category, and we define it exactly from the sieves. When we calculate Ω therefore, we construct this object, simply as always as a contravariant functor of a set etc., but we build it from the sieves on each of the objects in the category. In our case, there was a single object so it was very simple. It was very very simple.

So I was fascinated for a long time by the idea that Grothendieck had called sieve and that he was not unaware that this name had already been used by mathematicians, and that there is for example a sieve which is well known and which is the sieve of Eratosthenes. So I finally found the answer, I finally found why the sieve of Eratosthenes is a sieve, in the sense of Grothendieck and that, it comes from a common work we did with Katia Consani and in which the category we take is very similar to that from earlier, where there was only one transformation, but this time, it is a little more complicated anyway, because instead of having (we always have a single object, as before), but instead of having the powers of a single morphism, we have an action of the multiplicative integers. That is, for each integer, we have a morphism, and when we make the product of two integers, the morphisms are composed. So it is an exercise to demonstrate that the sieve of Eratosthenes is a sieve in the following way : it is very funny. Because... what is the sieve of Eratosthenes ? The sieve of Eratosthenes, that is to take the first non-trivial number. We're going to fuck 1, we don't care 1, okay. So we take the first non trivial number which is 2. And what does the sieve do ? The sieve considers all multiples of 2, all even numbers except 2. And then there are things, good. There are 3 left for example, then it takes all the multiples of 3 except 3. And then there are things, 4 we have already taken since... So it takes all the multiples of 5 except 5. Well, I pretend that if you look at the integers like the morphisms, multiplicative integers, such as the morphisms of a category that has only one object, and if you look at everything I just told you, i.e. if you look at all even integers except 2, all the multiples of 3 except 3, etc., that makes a sieve in the sense that I gave you earlier. And it shows you how subtle the notion of truth is for this category, because I only gave one example of a sieve. To check that it's a sieve is trivial, it's not the question, it's not the difficulty.

So now, once we have the notion of sieve, we will see the notion of Grothendieck topology. I couldn't give a talk on topos without giving the definition of a Grothendieck topology. So, I will tell you the moment which for me was crucial in the appreciation of the concept of topos. The crucial moment was this : before, when I was presented with a topos, they always presented me with a topos saying to myself "I take a category, a small category, and I suppose that it is stable by fiber product." At that time, my ear closed and I was thinking of something else, okay *(laughs)*. And the reason is this : it is that, when we say that, and after that we write what is a base, etc., we obviously have topological intuition in mind ; that is to say that when we say that the category has fiber products, we think of two open ones that have an intersection. And from there, good, we can develop things. And so what was crucial for me was when I understood in fact that, already in SGA4, Grothendieck had defined the sites, and the fiber products on the sites, *without any hypothesis* on the small category, *without any hypothesis* on the small category, we have absolutely no need to assume *anything* about the small category, and the huge advantage is that when we do that, we understand better what we are talking about. You know, in math, there's one thing you need to understand is that the main difficulty when you are faced with a problem is to manage **to think right**. And thinking right, it sounds silly, it looks like... trying to think right... but once we get there to think right, things fall like ripe fruit, but you have to know how to think right. And it's not just to think of asking the small category to have fiber products. Thinking right is to think what there is there, that is to take the maximum sieve, the fact that when you have a sieve... So, what is a Grothendieck topology, it is a collection of sieves, we give for each object a collection of sieves, and we have compatibility conditions. But what is the intuition that you need to have behind ? No matter the detail of the axioms. What is the... When you do topology, you have the intuition of open collections. This is a very delicate intuition, I will explain to you why it is very delicate. Take for example the interval $[0, 1]$. And then take in $[0, 1]$ only rational numbers. They are dense, so you will recognize the open ones, with the rational numbers, since the open ones are interval meetings. An interval, I know it by its intersection with the rational. Okay ? What will change ? Why is it that if I take the topos that is given by the rationals with these open ones, I get something different than the topos which is given by the interval $[0, 1]$ with its ordinary open sets ? They look alike, they seem to be the same. Well, if you search, you will find that there are actually a lot more openings for rational than there is for the real. For rationals, there are open covers that are there, while they are not there for the reals. Here. Typically, what you want is that if you take a continuation of larger and larger openings but whose limit is an irrational number, well, that's fine appear as an cover at the rational level but it will not be a cover at the real level. Okay ? That is to say that at the real level, if you take the complement of that, the combination of the two, it will not be an open overlay. So in fact, there are a lot less open collections for the real that there are for the rational. When we think topologically, we think like that. When we think at the topos level, we think differently : how do we think at the topos level ? We think that sieves mean small things, they mean small objects. Sieving will give objects that are small. And at that point, the axioms, they become almost absolute - clearly obvious. And what does it mean that an object is small compared to an open cover ? What does it mean that an overlay is small compared to an open overlay ? It means that it passes through, it means that it is contained in one of the opens of the covering : it passes through a hole. So this is the intuition that you need to have : the intuition of the sieve is that these are things that are small, and pass through the holes. Okay.

So having said that, now we have the intuition of a Grothendieck topology when there is a basis, etc. ; I'm not going to bother you with that. So there is an essential notion in topos but it is similar, I will not talk about it too long : it is the concept of point. And especially the notion of geometric morphism. So if you want, the topos... It happens that once you just think about topos, the same properties that are true for topological spaces continue to make sense, but obviously, they are much more subtle. Typically, what happens, and that, I copied a page of SGA4, this is what a morphism from one topos to another, what is called a geometric morphism.

So to understand what a geometric morphism is, that is to say a morphism from a topos in another, you have to have some familiarity with the sheaves on a space. Why ? Because when we have a continuous application from space X to space Y, well, there are two ways to connect sheaves on X with sheaves on Y. There are two ways to do this. And these two ways, there is one that is tautological, almost trivial, which consists in taking a sheaf on X and pushing it forward towards a sheaf on Y. And that, in what sense is it trivial ? It's trivial because it's enough when you take an open on Y, take its inverse image and look at the sections of the sheaf on X on this open, on the reverse image. So that makes a beam, there is no problem. So this definition, it will self. But there is another way to connect the sheaves of X and the sheaves of Y which goes in the other sense, that is to say it sends a sheaf on Y towards a sheaf on X, and that one is much more interesting, it is much less trivial. It's visually obvious if you think of a beam as a space spread over the base space, and this is particularly the case for sheaves of sets, but, where it is extremely interesting, is that this application that goes the other way, it has a wonderful property, it has a totally unexpected property. First, it is deputy on the left the other. That is true, it is not a big thing, we could have defined it like that. So it is the adjoint to the left of the other, of the one that went forward, very good. But it has a wonderful property, and this marvelous property is the property that it is exact on the left, that is to say that it commutes with limits. So this is an extremely powerful, extremely amazing property, and I think that the example that is due to Pierre, the most striking example of that, you have to be struck by an example, until you are struck by an example, you will not understand. The most striking example of that is what we call simplicial sets, simplicial complexes. So what you do is that there is a small category, therefore a little more complicated than that of earlier, *(intervention by Pierre Cartier : which Grothendieck does not want)*, taken over by Alain Connes, which Grothendieck does not want, precisely. I'm going back to Grothendieck's page because he doesn't want it. It's fun, by the way. Here.

It's the one Grothendieck doesn't want. This small category is called Δ^{op}, it is the category that is semi-simplicial ? These are the finite sets, completely ordered, with the not decreasing applications. This category is very important for the following reason : in topology, in years 40-50, a concept developed, at the beginning, it was formulated in a little too simple, which was the concept of simplicial complex. We took a space and we triangulated it. When we take ordinary space, we can triangulate it, or in a larger dimension, when we triangulate it, we can give a combinatorial data which encodes the triangulation. This combinatorial data, we can formulate it by looking at what is called the simplicial complex but in a fully combinatorial way, taking simplexes, etc. So it happens that if we do things like that, it doesn't work very well at all for the product. That is, since the product of two simplexes is not a simplex, by example, the product of two intervals, it's a square, it's not a simplex, but it doesn't work at all for the product. But it's because we didn't think right. And it's because we don't have done something, that seems trivial when you do it, but it is fundamental. And this thing which is trivial when you do it, but which is fundamental, is that you have to understand a lot better geometric realization of this combinatorial object, and this geometric realization of the combinatorial object, in fact, it is a point of a topos. It happens that this category is associated with a topos, the topos bébête, the topos of contravariant functors that go from this category to the category of sets, and that, it is a theorem that we can easily demonstrate, the points of this topos, in a direction on which we do not go to drag on, the points of this topos are exactly the intervals. That is to say, these are exactly the totally ordered sets which have a smaller element and a larger element. So the points of this topos are given exactly like that. And when we have a point in the topos, well, the inverse image functor, which goes to the sets, well, this functor is the geometric realization functor if we takes for totally ordered space with a smaller element and a larger element, if we take the interval [0 , 1], that gives exactly the geometric realization of the simplex, of the simplicial complex.

So now, wonder of wonders : this functor preserves finite limits and therefore, it preserves products. And so, when we take the stupid product of two simplicial sets, that is to say of two contravariant functors from this little category to the sets, well, when we take the geometric realization, it will give the product of geometric realizations. It's an immediate exercise to check that it is compatible with the topology. It does not present any difficulty, the difficulty, it is purely set designer. And, Pierre, it was you who demonstrated this theorem for the first time, right ? *(Pierre Cartier answers "Milnor")*. Yes, Milnor or you. But, what you have to see is that the concept of topos includes that thing. It understands this thing and generalizes it to an absolutely incredible point, that is to say that a point of a topos now, will precisely preserve not only the arbitrary colimits, but will preserve the finite limits, therefore will preserve the products, etc.

And that's why when we take a point from a topos, it takes us to set theory but respecting everything we know. That is to say that it will transform an abelian group in the topos into a real abelian group ; it will transform all the elementary notions that we can have into a real notion in set theory. So, there is one step that I will not dwell on at all, but which is extremely important and in which precisely, there are very very interesting works which is done now, which concerns the classifying topos. That is to say that exactly as there is a classifying space for bundles, or vector, etc., there is a classifying topos for logical notions. And one of wonders of that, which answers a little bit to Grothendieck's question when he says *"the age-old category* Δ^{op}*"* is that the topos which is associated... not directly this category, but the topos which is associated with this category, it is exactly the topos which classifies the intervals. That is to say that if we define abstractly what I explained earlier, that is to say an interval, a completely ordered set, but we should not speaking of the whole, in an arbitrary theory, well we see that this notion has a classifying topos and that this classifying topos is exactly the dual of the category Δ^{op}. Well.

So we're not going to go into details. Now we're going to do something else : I don't want to go back in the technical details, I don't want to. We're going to come back to Grothendieck, we're going to read again Grothendieck and then we will finish by reading the end of the exchange between Grothendieck and Serre in their correspondence. So that's what Grothendieck says. Well, it's very important to have talked about topos, but it's still more important to try to perceive Grothendieck's way of working, because that's what we need. Okay, sure, maybe we will use topos to do all kinds of things, but we also need, terribly, in our civilization : when we are now witnessing a speech that is done in public, we realize that there is a third of people who have their computers open in front of them and who do their emails, *(laughs)*, or who do something else, or cell phones. But it's a disaster, because when you read Grothendieck and when you immerse yourself in his way of thinking, you notice a thing, the most striking thing is the time he had. One has the impression that he had an infinite time, infinite time, that he was not constantly disturbed. You know, now, we're talking of generation Y, that is to say people who do 3 things at once. We believe we save time, but it's not true. We now have a basic need in our civilization to isolate ourselves, and to be able to think slowly, and to take the time to check everything, to be sure, to do it twice, to do it three times, etc. That's why I made it last, when Grothendieck was talking about sheaves, it lasted, *(funny)*, it lasted, but it was on purpose that I did it, I did it on purpose, because I wanted you to realize this fundamental slowness. It is a slowness which, when you feel it in the first degree, is irritating. It's the slowness of the turtle and the hare, if you will *(laughs)*. And it is she who wins. So here's what Grothendieck says :

"When I'm curious about something, mathematical or otherwise, I question it. I question it, without worrying if my question is perhaps stupid or if it will seem such, without it being at all costs maturely weighing. Often the question takes the form of an affirmation - an affirmation which, in truth, is a blow probe. I believe it more or less, to my assertion, it depends of course on the point where I am in the understanding of the things I'm looking at. Often, especially at the start of a search, the affirmation is downright false - it still had to be done to be able to be convinced. Often it was enough to write it down." .

One basic thing that Grothendieck often does is that he is able to write an idea that is not yet ripe. He is able to start writing, that's a fantastic quality.

"Often, it was enough to write it so that it was obvious that it was wrong, whereas before writing it there was a blur, like a malaise, instead of this obviousness. Now you can go back to the burden with this ignorance less, with a question-affirmation perhaps a little less "next to the plate". More often still, the statement taken literally turns out to be false, but the intuition which, awkwardly yet, tried to express itself through it is just, while remaining blurred."

I stop for a second : when he talks about writing, the computer is still a disaster, because we write better, in this kind of situation when writing on paper with a pencil, because when you write on the computer, it has to look perfect. We are going to ask ourselves questions of LaTex, we are going to ask ourselves questions like that, but it's completely ridiculous, we're not there yet, we're at a point where we want to leave the pencil that does what it wants on the sheet of paper. This is very very important. So this is what he says :

"This intuition will gradually settle down from a gangue just as shapeless at first of false ideas or inadequate, it will gradually come out of the limbo of the misunderstood that is just waiting to be understood, the unknown who only asks to let himself be known, to take a form that is only hers, refine and sharpen its contours, as the questions I ask of these things in front of me arise more precise or more relevant, to define them more and more closely. But it also happens that by this gait, the repeated soundings of the probe converge towards a certain image of the situation,..."

That means that we are in the process of forming a mental image.

"... coming out of the mists with features marked enough to lead to a beginning of conviction that this

image there expresses well reality - when it is not however, when this image is tainted with an error of size, likely to distort it deeply. The work, sometimes laborious, which leads to sieve of such a misconception from the first "takeoffs" noted between the image obtained and certain patent facts, or between this image and others who also had our confidence".

It must be said there, that it is very good, in these cases that he describes, to take a step back, to do other thing, and Grothendieck often had, Cartier often told me that he had 100 irons in the fire. When we see that things tend to mess around a little bit, it's better to take the field, because in fact, we are viscerally attached to the ideas we had, and we don't want to accept that they are false.

"This work is often marked by increasing tension, as we approach the knot of contradiction, which at first becomes more and more glaring - until finally it breaks out, with the discovery of error and the collapse of a certain vision of things, arising as immense relief, like liberation. The discovery of the error is one of the crucial moments, a creative moment among all, in any work of discovery, whether it is a mathematical work, or a work of self-discovery. It's a time when our knowledge of the thing suddenly probed renews itself."

And now here is one of the most magnificent paragraphs I know :

"To fear error and to fear the truth is one and the same thing. Whoever is afraid of making a mistake is powerless to discover. It is when we fear to be mistaken that the error that is in us is made immutable like a rock. Because in our fear, we hold on to what we have decreed "True" one day, or what has always been presented to us as such. When we are ripe, not by the fear of seeing an illusory security vanish, but by a thirst for knowledge, then error, like suffering or sadness, crosses us without ever freezing, and the trace of its passage is a renewed knowledge.".

If one day, you are not in the mood or all that, read that sentence again. It is a kind of talisman.

So I'm going to finish in... I started with the discussion between Serre and Grothendieck, at the very beginning, on Grothendieck's Tohoku article, and I'm going to close with a rather different note, in a very different tone, which is precisely the reaction of Serre when he received *Crops and Sowing*. So, good, I don't know if you know Serre, but I mean, he's not used to mince words, and he don't really like moods in general and so, I mean, it's extremely interesting that in the correspondence between Serre and Grothendieck, they continued their exchanges, at the time when Grothendieck had deliberately isolated himself from the mathematical world. I mean, it's not the mathematical world that had chased him, it was Grothendieck who chased himself, who isolated himself from the mathematical world, he wrote this text ; all the passages that I read to you from Grothendieck are in *Crops and Sowing*, so it's an admirable text, and you have to read it with a certain perspective, of course, because there are times when, if you like, he says things that are not ideal, but in any case, he expresses himself. So here's what Serre says after receiving it :

"Dear Grothendieck,
I have received the Crops and sowing *leaflet that you sent me. Thank you so much. The penultimate booklet is still missing to me, booklet of which I only have a few isolated pages." (laughs)*

Well obviously, there are so many pages. I must tell you, moreover, that it is a text that you must read... you must not read more than 5 pages at a time. I remember spending an amazing summer by reading in parallel *Crops and Sowing* and Proust's *In search of lost time*. And I mean, from the same way, that is to say, of course, people who are looking for crisp anecdotes, they will read by skipping pages, if you do that, you lose everything. It's exactly the same with Proust. Proust, you can't read it by reading more than 5 pages at a time, you have to meditate on it, you have to rethink to it, etc. It is necessary to let yourself be penetrated by an atmosphere that is absolutely extraordinary. So this is what Serre says, he says : *"One thing strikes me. In the texts that I could see, you are surprised and you are indignant at what your alumni did not continue the work that you had undertaken and largely carried out. But you don't ask yourself the most obvious question, one that every reader expects you to answer : "Why did you abandon the work in question ?" " (laughs).*

It is still quite a question. And then what is great is that Serre has an answer, and it's not an obvious answer at all. No, no, but it's a letter from Serre but he continues his letter and he has a proposition as to why Grothendieck left. So this is what he says ; he says :

"I have the impression that despite your well known energy, you were just tired of the enormous work that you had undertaken."

Anyway, everyone will understand. I mean when I talked about the time, it means he had little time to do something else. So I mean, it's huge. At the beginning, I read you passages in which he talked about everything he had to absorb, etc., well I mean, it's monstrous such an amount of work.

"Especially since there were also SGAs who were falling behind, year after year. I remember, in particular, the rather disastrous state of SGA5 where the editors got lost in masses of diagrams, whose commutativity they were reduced to assert without proof, to the nearest sign by being optimists." (laughs)

"And these commutativities were essential for the future. It's in this disastrous and not idyllic state, as you would think to read Crops and Sowing *that my sentence from the Bourbaki seminar refers to, the version definitive of SGA5 which should be more convincing than the existing presentations and handouts."*

It's Serre spit.

"We would like to have your impressions on all this, even modified by 15 years of burial for borrowing your terms, we're hungry."

So now he's going to go to a much deeper explanation :

"We can wonder for example if there is not a deeper explanation than simple tiredness to have to carry so many thousands of pages at arm's length. You describe your approach to math somewhere, where you don't attack a problem head-on, but where you wrap it up and dissolve it in a rising tide of general theory."

This is what I was talking about earlier when I was talking about thinking right. And for example, there is an anecdote, Cartier will not contradict me, which is that once, going up from the cafeteria to the IHES, there was... I believe it is Demazure who is asking Grothendieck a question on $SL(\mathbb{Z})$ or on... So what Grothendieck says that this is not the right way to formulate this question and the result was SGA3, i.e. Grothendieck's theory of algebraic groups *(laughs)*. So that's Grothendieck, it's... He can have a specific question, we can ask him a specific question, but he will say "this question is not in the right context". And he's going to develop a general theory to let the question become natural. And from the moment when the question is natural, and when we took the pain and time to think right, it will fall like a ripe fruit. So that's what Serre says when he says :

"...but where we envelop and dissolve it in a rising tide of general theory."

So the question dissolves. And in *Crops and Sowing* elsewhere, Grothendieck has very beautiful pictures, he talks about a nut, and he says that there are two ways to deal with the nut : the first way, is to take a hammer and break it, and the second way is to let it soften in water, etc., so that eventually it opens on its own.

"Very good. It's the way you work and what you've done shows that it works, less for EVT[5] *and algebraic geometry."*

So that's what Serre says, and that's serious. He says :

"It is much less clear for number theory, where the structures at play are far from obvious, or rather, where all possible structures are at stake."

And I prefer to finish on this. That is to say, if you will, it's extremely striking to see these two ways of thinking about mathematics. Grothendieck's way, okay, which is a way which is precisely to try to think right, and to try to formulate, if a problem is given, to formulate it so that it falls on its own, and if you want, explore every corner, every single one nooks. In his house, the house of which he speaks, there is no corner which is dirty, which is not explored, etc. He wants everything to be flawless. And he can only think when it is like that. And the price to pay, it's a colossal job. But it's a job that is not really difficult, in the sense that, we develop things, etc., etc. At no given time if you want, we are on a steep

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cliff, and we risk fall, at none of these times. It's kind of like you know Israel, like how the Romans wanted to attack Masada, I don't know if you know. Okay, this is something very striking because they backfilled with earth, earth, earth, so that it finally happens to the level... and it took them, I don't know, I think it's ten years or something like it *(The public gives their opinion, 3 or 4 years).*

That's the Grothendieck method. And what you see in hindsight is all you can learn from this method. All we can learn... As opposed to another method, that I like a lot, which consists of, in the corridors of the École Normale, in time, when I was at the school, there was a friend who had posed a problem to me : he was on the third floor and I was on the ground floor, and then I went for a weekend, and then I spent my whole weekend trying to solve... good. This is the problem solver, if you want, we give you a problem, and you are looking for to solve it, well, you're looking to solve it in the most efficient possible way. These are two completely orthogonal ways of acting and in fact Grothendieck has a whole discussion in *Crops and sowing* on these two ways of acting and he distinguishes them, well, he formulates them with yin and yang. Well, but it's very very important : he says that the method he has is more feminine, if you will, the other, which is a male method. It's hard to say exactly. But it's very important, when we do math, to imbibe this idea, actually, that there is this need, and that often, we don't believe it. For example, recently, I had a colleague who had posed a problem for me at the Academy that I ended up solving, but I was flabbergasted that I solved it when I started to think right. It amazed me ! Because I would be told "but you want to solve a problem, but why do you worry about that ?". No, that's not true, it's something fundamental, getting to think right, it is something absolutely fundamental. It will never be useless to try to think right. Never it won't be useless, okay.

So I hope I made you want to read *Crops and Sowing* and then, above all, to manipulate the simplest topos, and try to use them, compared to our logic, which is very poor, even in circumstances completely outside of mathematics. Obviously, it requires work, it requires work that is very slow, etc. which is that of appropriating the concept. And it's a notion which is cursed, it is cursed : with Pierre, and then especially with Laurent Lafforgue, for example, we tried for several years to support a very very brilliant mathematician, who is Olivia Caramello, and we encountered hostility, not to say contempt, from the mathematical world in general. And we were able to experience on this occasion how there is a kind of, I don't know, fatality, on the notion of topos, there is something that irritates people, because probably, they feel, this is what Grothendieck says, he says it so well, he says it explicitly, he had already felt it in his time, no doubt, they feel that there is something, but they do not really understand it. And to really understand, you have to do it, of course, but there will come a time when the concept will belong to you and you will come to appropriate it. And the best way is this metaphor is that space is not front of the stage, it is behind, it is a kind of Deus ex machina, and it's him which makes the sets rotate, it's him who introduces a hazard, a hazard in the sets, in set theory. Just as there is a hazard in prime numbers, which we all know, and even that there is a quantum random, so here, we must keep all this in mind, and I will stop there.