The journey of a mathematician
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So, so I apologize in advance for the narcissistic side of my presentation. But I mean, it's part of the game and hey, I was asked to talk about my journey. And so, I'm going to explain a few things that you won't find in other conferences and that you will not find anywhere. The first scene, if you want, it performs at Lycée Thiers, so in Marseille in the year 1966, and it's in May and I'm taking the exam of École Normale Supérieure. And this is the first test. A test which lasts six hours. I am sitting on a bench and I have an immediate neighbor. We are given the math problem and my neighbor, he begins to write, to write, to scrape. I read the problem statement. And then, an hour passes and my neighbor continues to write. I do not understand the statement of the problem. It's not coming... I mean. After two hours, I look by the window and I think about nothing. My neighbor writes. Three hours, it lasts six hours, three hours, nothing. Four hours. 5. Nothing. 6 hours. I leave the room in practically making white copy and leaving the room, I find the solution of the problem. Well, then in such a case, normally, the conclusion is clear. But I had a group of great friends, I had a group of extraordinary friends. They told me "You can't do that, you can't stop, we're going to go and swim in Cassis!", so they take me to swim instead of going home and moping. They take me to bathe in Cassis. We bathe and all that, I start to relax, all that. Then, the next day, I say "I'm going back". Go on hop! Then I was received at the École Normale. So if you want, what I thought about before I made this conference, it's trying to give you tips that can really serve you. So when we talk about tenacity, what does that mean? Tenacity doesn't that we are stubborn, etc. No, it means that when the circumstances are the worst they could be (and there were, I mean the main test, 6 hours, nothing) so, when the circumstances are the worst they can be, eh well, don't give up. When we can continue, we must. We must continue. Here. That's what happened at the beginning, at the very beginning.

And so, I went back to École Normale Supérieure and there, I found an absolutely extraordinary atmosphere, that is to say that at the time, we were not forced to nothing, we were not forced to do anything and in particular, we were not forced to pass the aggreg. And what mattered was simply to ask each other problems and to try to solve them, etc. That was the atmosphere of the School and at the end, three or four years later, I had a teacher who was Gustave Choquet and I had been duit by a theory called standard analysis, which still exists. But I hadn't realized when I was seduced by this theory that in fact it was something that was a bit chimerical. And my teacher Gustave Choquet had sent me to a summer school of physics which still exists, the summer school of Les Houches, which was created by Cécile de Witt

Lecture given as part of the Maths for All cycle on 18.12.2017 at the École Normale Supérieure, in Paris, viewable here https://www.youtube.com/watch?v=QfZLKxKTS2c and which is an extremely interesting place for the communication, precisely, of more experienced people with students. And so, well, I'm going to listen to lessons at Les Houches Summer School and I was spotted at this, then, and I was sent the following year... I was invited to Seattle, to United States, at the Battelle Institute, and I was invited there. I was young married and I had decided to accept the invitation. It was mainly to visit the United States. And then, with my wife, we went to see my brother Bernard, who was at that time in Princeton. It was July and in Princeton, the weather was terrible. But you know, the humid heat, and there was only one place on campus that was nice. It was the Book Store. So with Danye, my wife, we spent the afternoon inside, at the Book Store. Well, at the time, I didn't have much internet. There was really very interesting books and I was looking for a book because I knew that we had decided to cross Canada by train. Instead of flying to go from Princeton, from New York to Seattle, we decided to go to Canada, and to cross all of Canada by train. But it took five days. I said to myself five days with nothing. It will do a lot. I will try to find a book from math and then read it during the trip, I hesitated a lot. I hadn't looked a lot of books, I hesitated between a lot of books. I ended up taking this one, and I had looked at it, I had tried to understand it, I did not understand everything. I had tried to understand it during the Canadian train ride. There were great plains that we crossed for days and days and days. And then, we got to Vancouver and Victoria, Victoria Island and finally we got to Seattle after a week.

And in Seattle, when we arrive, I went to the Battelle Institute to watch the conference program. And to my amazement, I found that the author of the book, which I bought completely by chance when I went to Princeton was lecturing there. At that time, I thought... Ah really, if you want, it's something completely extraordinary happening, so in fact, I will not listen to any other conference. I'll only go to this one. They were conferences on von Neumann algebras, my first subject and so here, von Neumann, it is known above all for something other than von Neumann algebras. But it's him, if you will, who developed something called operator algebras. And the theory's conceptor that was behind the book I bought, it's a Japanese mathematician called Tomita. And in history, it is also a completely extraordinary, so I continue to tell you stories. I hope it's OK.

Tomita, if you will, is a Japanese mathematician who miraculously escaped the war between Japan and the United States. He was deaf since the age of two and as he was deaf, in the regiment in which he was, he was exempted from going to the expedition that had to be made, because he was deaf, he couldn't hear well orders. They left him alone. The whole regiment was wiped out in the former that they made. He found an absolutely brilliant theory, but he had a hard time communicating and it's another Japanese mathematician called Takesaki, who wrote a book in those years on the theory of Tomita, the theory that Tomita had invented. And so, if you will, what happened is that I listened to Takesaki's lectures on the theory of Tomita and so that was a fairly new theory, etc. And when I returned to France. I decided to go to Jacques Dixmier's seminar. So, if you like, Jacques Dixmier had a seminar on operator algebras, operator algebras that had been invented by von Neumann. They were invented to understand quantum mechanics, to understand what are called subsystems.

In quantum mechanics, therefore, there was a formalism of quantum mechanics which had been well developed and von Neumann wanted to understand the subsystems and there are subsystems which correspond to factorizations of the Hilbert space. But in fact, von Neumann, with Murray, had discovered other factorizations and he had discovered three types of algebras called von Neumann algebras. There are what are called type I algebras which are very simple. There are type II, which are much more incomprehensible, and type III were the others, those that were left.

So when I got home, there was the Dixmier seminar. So I went for the first time at the Dixmier seminar and in the Dixmier seminar, Dixmier proposed a subject which was the classification of algebras and he distributed articles that then, we had to exhibit in his seminar to explain to others. So there too, I have raised my hand to get an item, I went to get the paper and when I got home in the suburban train, something completely became obvious to me, that was that the article he gave me to expose it in the Dixmier seminar, which was a priori on a totally different subject, was in fact perfectly connected to the theory of Tomita. And that was the start of my thesis. So, in fact, what I did, I wrote a small letter to Jacques Dixmier. He said "Okay, your letter is only half a page, it is not detailed enough, etc. Send me a more detailed letter.". I sent him a more detailed letter. I went to see him in his office and the only thing that he said to me when I went to see him, he said to me "Go for it !". And from there, the things went naturally but there was a kind of competition circumstances that made it happen, at least in the beginning, absolutely wonderful. Then, afterwards, there was a period, of course, when I had to do extremely complicated calculations, etc. And there was a moment when there have been... if you will, in a lifetime, there are very few moments like that. There may be two or three at most. There was a day when I brought Danye at his high school and I was driving home and at one point there was a red fire. So there, I realized, my brain realized that there had one thing that was completely obvious, that was before me, that I had not seen before, and which made it possible to completely unblock the situation. What was it ? It was the following. It was that Tomita and Takesaki, therefore, had demonstrated, if you will, that if we have an algebra like that, which we call a von Neumann algebra. If we give a state of this algebra, it's a rather complicated thing, we automatically have a group with one parameter of automorphisms of this algebra. But what was absolutely unclear was that it depended on a state of the algebra. What does it happen when we change the state of algebra ? What I understood when I was at the red light, it was because when we change the state, the group with one parameter practically does not change.

It does not change. If you want. If we neglect what are called inner automorphisms which are the automorphisms which appear naturally when we take a non-commutative space, when we take a non-commutative algebra, automatically, the automorphisms that come from non-commutativity, when they are cleared up, the ambiguity disappears. So what was, if you want, the message, what was the philosophical consequence of this thing. It's something that followed me all my mathematical existence. It is the fact that non-commutativity implies, generates time, implies a temporal evolution. So, I did not make a transparency on that, but to explain what non-commutativity is, you have to remember that. I'm telling you again a story that explains how it was discovered in physics. It was discovered in physics by Heisenberg. So Heisenberg was a physicist and at one point he was, I believe, in Göttingen and he was victim of a hay fever that was terrible and at that time, they could not treat him effectively. The only way to cure hay fever was to send people to an island where there was no pollen. They sent him to an island called Heligoland and hey, so he was on this island, he was housed by an old lady and there, he got down to do calculations. He was doing calculations and towards 4 am, he had an illumination. He understood. In fact, he saw a species of landscape that was revealed before his eyes. He understood that the physical quantities, for example, if you write $e = mc^2$ or put c^2 times m, it does the same.

Heisenberg understood that when we look at a microscopic system, it is not the same. That is, if you multiply the position of a particle by the moment or the moment by the position, you don't get the same result. And that was something huge he found. And what he says in his memoirs, he says that instead of going to bed when he made this discovery, he could not. He went to climb a rocky outcrop which was on the edge of the island and he waited for sunrise of the sun at the top of a rocky peak. So if you want, the extraordinary thing, is that the physical quantities at the microscopic level do not switch. That's it which completely unlocked quantum mechanics and that's what resulted in von Neumann to develop von Neumann algebras because von Neumann algebras are precisely, if you like, the algebras that are going to fit into quantum mechanics. So what ? So the contribution that I made in my thesis, if you want, it was to understand, precisely, that in fact this non-commutativity, it will generate time in a completely canonical, completely natural way. So that, of course, gave a bunch of invariants for these algebras. It allowed to classify them, but of course, it took a long time to classify them, i.e. that it also solved type III. It helped reduce it. That's what I did in the second part of my thesis, it is to reduce type III to type II with automorphisms. But of course, afterwards, we had to classify types II at least in the hyperfinite case and classifying automorphisms, it took, it took me an absolutely considerable duration and there was a period of my mathematical existence which was very, very conducive for that. It was the period of my military service. So of course, you will laugh because if I had really done my military service, that would have been very difficult to do math. But hey, I was lucky. I did my military service in cooperation with underdeveloped countries, with English Canada...

So it was still pretty cool. So I was in a small university which is the University of Kingston, Ontario. And there, again, I want to say, we have met extremely extraordinary people, humanly, around us, in a small group. And the fact that this university was not a central university, it was not Princeton, etc. It gave freedom of thought which is incredible, that is to say if you want, it helps in a small university, or if you are in a place like that, to be a bit offbeat. It allows not to have the weight of science, of all knowledge, etc., on the shoulders, and that allows to have some freedom. So I felt this freedom maybe more than ever in that place. So I was very happy at the time since I had practically finished what I wanted to do which was to understand the factors, to understand type III, etc.

And then, I returned to France and when I returned to France, I was invited at the IHES, at the Institute for Advanced Scientific Studies. And there, when I got there, I had a hell of a shock, that is to say that I went to lunch where people go to lunch usually, there is a small cafeteria and there were people talking about math. I had absolutely no idea about what they were saying. That is to say, if you want, I was a specialist in a sharp subject, of course very difficult, but I was not... I did not know, for example, what it is like a De Rham complex, etc. There was all this stuff going on over my head and I got lucky again. I was really lucky. I met a mathematician named Dennis Sullivan who, at the time, was one of the pillars of the IHES. That was in 1976. And this mathematician, he had the following property, very, very unique. He had the following property, which was that when he saw someone new at IHES, he would come and sit next to him, and he would start asking questions, with "What are you working on ?". So, we were chatting with him and at the start, we thought he was completely silly, because he was asking naive questions, if you want. Then he went on, like this, and he kept asking questions, of a naive nature. And then, after a dozen of questions, you realized that you didn't understand what you were talking about.

He had an absolutely extraordinary Socratic power. So, I started to discuss, discuss in detail with him, etc. And by talking to him, he taught me a lots of things, he taught me a lot of things I didn't know. He taught me the geometry and what bothered me a lot was that the work I had done on von Neumann algebras, if you will, it seemed to be in a place quite out of the center of mathematics. There is a kind of mathematical landscape and in this landscape, there are places that are really quite in the center. And then there are places that are much more peripheral and I had the impression that it was related to physics, of course, but I had the impression that the von Neumann algebras were something that remained quite peripheral. And I realized, talking to Dennis Sullivan, that in fact you could associate with a geometric datum which is perfectly known, which is what we call shovel a foliation, you could associate it with a von Neumann algebra. And what did it work ? It allowed to illustrate the classification I had made from geometric objects that were perfectly understandable geometric objects. The simplest foliation, you know, foliation, you have thought of a reverse lamination. It's basically. It is a space like here, the torus. But the leaves of the foliation, these are the lines that wrap around here. But what is striking in a flipping is the fact that while the total space is compact and finite, the leaves there in the winding, are infinite, that is to say that the leaves, when looked locally like on the right, we see something that is very simple, it's a product. But when you look at the leaves overall, they don't come back at the same location. They wind up endlessly, okay. So it was a point absolutely crucial because the foliation, in geometry, people know very well what it is. And they have lots of examples. And it turns out that the classification that I had done with the factors of type III_2, type III, etc., which seemed something rather odd and quite remote, fairly off-center in the mathematical landscape, in fact, it was perfectly illustrated by the most natural object that you can imagine. For example, this page is a sheet-floor of Type II. And if we take for example a geodesic and cyclic foliation, it is a type III foliation, etc. So if you want, that, that gave sense to the algebraic objects that I had found, it gave them a geometric meaning. Another meeting that was absolutely crucial to me at that time was in 78, I was invited to give an hour-long lecture at the International Congress of mathematicians. I was very struck by the following thing, that when I did my presentation, I had prepared and prepared, prepared. When I made my presentation, of course, my talk was about my results, about the classification of factors. But at the end, I had added results on the foliations, an appendix, and then so very, very odd, for me, the part on the leaves of a foliation, it's a complete trivial part and on the other hand, the really really hard, very, very hard part was the part on operator algebras. After my presentation, Armand Borel came to see me and he was all excited about the part on the foliations.

And suddenly, he invited me to Princeton and I was invited to Princeton during the year 78-79 and there, at Princeton, I had a meeting that was going to play an essential role in my mathematical life. It is the meeting of my collaborator who is called Henri Moscovici, who is a professor at Colombus, in the United States, and with whom, if you want, I really collaborated, most of my articles were written collaboratively with him. I also need to tell you another story about Princeton. I had a colleague at the École Normale who had stayed at Princeton before, for a very long time. I think he had stayed while we were still students at the École Normale. And he was quite fat. And while he was a little round, he had passed a year in Bristol and when he came back, he was absolutely skinny. We had asked him "But what happened ?". He told us that he had spent a year in Princeton, that he hadn't spoken to anyone, that he hadn't spoken to anyone, that is to say that it was a place that was so, how to say, prioritized, etc., that in fact, he hadn't managed to speak to anyone. So there was that side, there was that side, really, in Princeton. And I had an incredible chance which was to meet Henri and with Henri, of course, we immediately started working and we collaborated together.

So, if you will, what key point has emerged so far ? What was the essential point ? In fact, I understood at that time, because of the leafing through, I understood the scope of what was going on, because in fact what is going on, if you will, when you look at this type of space, what happens is that we actually have a space which, if we try to find its cardinality at the set level... If we look the leaves space of a foliation, okay, if we look at the space of the orbits here, if we look at the space of the leaves of a foliation, we will see that if we take the theory of ordinary sets, this set has the cardinality of the continuous. But we cannot biject it with the continuous in a constructive way.

In fact, we realize that it is impossible, we can demonstrate it, it is impossible to build an injection from this set into real numbers. So, in fact, I saw very, very gradually that these spaces in fact, if we tried to understand them in the usual way with the theory of functions, etc., we would not get them, absolutely not, and that the only way to understand them was to associate to them a non-commutative algebra. And this is the beginning of non-commutative geometry. Why is this the start, the very beginning ? Because the algebra of functions associated with such a space sees this space only at the level of the theory of measurement.

Now, measurement theory is an extremely fuzzy theory that allows you to tear the space into parts, etc. But who doesn't give the topology, who doesn't give everything else. And gradually, non-commutative geometry, it's a theory which allowed, if you will, to redefine, to reconstruct all the concepts which range from the theory of measurement, of course, to topology, geometry, differential geometry and even real geometry, Riemannian geometry. If you want, in the commutative case, spaces like that, in fact, what came to light at that moment was that there was a whole new universe, of completely new spaces, just waiting to be understood. But at that time, it was first necessary to know that it would necessarily be interesting.

Why were we sure it would be interesting ? We were sure it would be interesting because such a space was automatically a dynamic object. That is to say, such a space, automatically, has its own time, it generates its own time, while an ordinary space, like a variety, etc., it is static, it doesn't move. So these non-commutative spaces rotate over time. And this is something absolutely extraordinary. So we knew it would be quite extraordinary. But hey, well sure, afterwards, it was necessary to develop the theory and therefore it was necessary to develop the geometry. If you want, spaces in the fields, they do not commute. The first example, of course, this was the example that Heisenberg had found, that is, the example of the Quantum mechanics. So, we had to completely find, redefine the geometry for these spaces.

So when we talk about geometry, of course, well, in mathematics, there are all kinds of geometries that people have invented and that are more or less elaborated.

But hey, the most, the most relevant, the most important geometry is the geometry of the space in which we live. So, in fact, that's what will interest me, it has interested me for years. It is the geometry of the space in which we live. And what is quite amazing is that in fact this work is based on the quantum mechanics, it is based on quantum formalism, etc. In fact, I answer to a question that was asked by Riemann in his inaugural lesson. In his lesson, Riemann was fully aware that the notion of geometry that he had formulated, from Gauss, etc., from what we knew at the time, the notion that Riemann had formulated was not necessarily a notion that would continue to make sense in the infinitely small. What does it mean ? It was clear in his day that it covered great distances, but Riemann was extremely cautious. And what he explains is that the reasons why he doesn't not believe that it continues to have a meaning in the infinitely small, it is that the concept of solid body or the concept of light ray has no meaning in the infinitely small. In fact, we are immediately in the quantum domain when we look at this. So he explicitly wrote in his writings that in the infinitely small, it is all quite the same as if the concept of geometry will not conform to what it is, at the one he gave in his inaugural lesson. So, in fact, what is happening, so, of course he continues and continues. And he also explains that in fact, the founder metric relationships must be sought in the bonding forces that act in the space. I don't know how he got this intuition absolutely extraordinary. So what Riemann says, if you like, is that to understand really the geometry of the space in which we live, we must in fact understand the forces that hold things together. So, of course, since Riemann, there have been absolutely extraordinary progress compared to what Riemann said, in that, of course, there must be non-commutativity.

This leads directly to the domain which is physics, which is not part of math. But he explains how crucial mathematical thinking is for that.

And what is very important, above all, is that Riemann refers to Newton. And Newton had also sensed that when we go into much smaller distances that cannot be seen with the eye, there will surely be new forces that will appear. So what Newton says is that the pull of gravity or magnetism, and electricity are visible from a great distance. So we can observe in the usual way. But of course, we know everything that will happen at those distances.

The much shorter distances escape observation. And there is a book that I recommend to you on the history of particle physics whose author is Abraham Pais and in which he explains precisely how, in 1895, it was after Riemann, since Riemann was in the 1860s. Between 1895 and now, we succeeded at the level of vision in the infinitely small, to increase vision by a factor 10. It's a colossal thing and by doing that, in fact, the real microscope, the real microscope which allowed to see in this very small distance, it is the LHC.

Okay ? It was at the LHC, in fact, that we managed to pierce the structure to a level much smaller. But when we talk about much smaller distances, it is like saying much larger energies. So now, effectively, we arrive at 10 TeV, that is to say at 10 at the 13 electronvolts.

So what happened was that the formalism that I had to develop for purely mathematical reasons, to make the geometry of non-commutative spaces, proved, this formalism, in the 85s, from the moment when I returned to the Collège de France, it proved to be incredibly suitable to take the geometric structure of the space in which we live. But from experimental results, that is to say to come to understand that space in what we live, it is not simply the continuous at all scales, that it has a fine structure, but this fine structure, in fact, it is exactly, in the words of Riemann, dictated by the forces acting in the infinitely small.

So how did the paradigm change ? I can perfectly explain it. The paradigm has changed in the following way. So the purpose of the trip ?

If you will, what we have achieved in the very recent years is to take this kind of huge mechanism called the standard model coupled with gravitation. But understand it as just gravitation, lie on a space which is more subtle, which is more complicated and which has a structure finer than that of ordinary space. But then, what happens at the level of concepts ? Conceptually, what's going on is something very simple to understand. At the time of Riemann, at the time when Gauss, etc., defined their metric, the distance measurements were made while trying to take the shortest path from a point A to a point B.

This is what is shown here in this picture and in fact, there was a whole expedition which was made at the end of the Revolution and then until the 1799s, by two French men here. I don't know if you've heard of this, but they are the ones who measured the Meridian. They are the ones who tried to define the unit of length by measuring the distance between Dunkirk and Barcelona and from their measurements, well, there were all kinds of episodes, but from the measurement, we defined a unit of length which we called the meter.

And when I went to school, we learned that the unit of length was the meter-stallion which was deposited at the Pavillon de Breteuil, near Paris. It was a bar of platinum. But things have changed. It is the old geometry which consists in measuring lengths like that.

But what happened, something extraordinary happened which is that one day there was a meeting of the weights and measures conference. And there is someone in the room who said “your length unit, well, its length changes.”. It’s annoying anyway if the unit of length changes in length. And what had happened ? What had happened was that the man in question had measured the standard meter by comparing it to the wavelength of krypton.

And he realized that the length was changing. Little by little, physicists have thought about it a lot and they came to define the unit of length either as being the standard meter which is deposited somewhere at the Pavillon de Breteuil, etc. It is necessary to think well that when you say the unit of length, it is the standard meter deposited at the Pavillon de Breteuil, if you want to unify the metric system in the galaxy, if you explain to people on another planet in our solar system that only for measuring their bed, they must come to the Pavillon de Breteuil, it will be a bit complicated, so they found a much better solution.

They found a much better solution which was to define the unit of length. First, they took it from wavelength, krypton, etc. Then they defined from the wavelength of what is called the hyperfine transition of the cesium. Cesium has a certain hyperfine transition in the wavelength. This is a microwave type wavelength which is of the order of 3.5 cm and this allows to measure very, very effectively.

So what ? Obviously, that changes everything. Because if, for example, we defined the unit of length from the spectrum of hydrogen, for example, hydrogen is present throughout the universe. So there, it would be perfectly valid. Okay. So, it turns out that the transition between the old definition of the meter located in Breteuil and the definition from the wavelength of the cesium spectrum, this is exactly the transition between old geometry and non-commutative geometry.

It’s exactly that.

It is a spectral type geometry, spectral in nature. And so, that’s it, the change is the change in the unit of length. So, it’s a spectral geometry in nature and in addition, if you want, there is non-commutativity of algebra, it is practically imposed by what are called the gauge theories. Physicists have discovered strong interactions, for example, they discovered that there is not only electrodynamics, but that there are also strong interactions that hold quarks together in an atom.

And for that, they needed non-Abelian gauge theories. Well it is this that is really at the root of the fact that space has a very small structure, a fine texture which is non-commutative.

So there is a saga, a very, very long saga that I do not want to tell you. But

I'll just tell you the end, there have been ups and downs, that is to say I had collaborators, like Chamseddine, with whom we therefore made a model, if you will, a model of space-time that was based on these ideas, on the hyperfine structure, on the structure which comes from non-commutative geometry. It has had its ups and downs.

There was, in 98, the discovery of the neutrino, the mixture of neutrinos, the models of Calabi-Yau. So there, we gave up, we gave up for a number of years. Then we came back in 2005 with a new idea which was to change a dimension. Everything worked. Great. Except that in 2008, there was an exclusion of the mass of the Higgs who contradicted our work.

For example, I wrote a blog, I wrote quoting Lucrèce who talks about people who rejoice themselves in the misfortune of others. So indeed, theren we were very unhappy. And then, there was a period, so that, that was from 2008. And then, there was a kind of resurrection again, simply, it's the reason I'm telling you that you should never get discouraged.

Never be discouraged.

There was a very long period of discouragement from 2008 to 2011 and in 2012, my collaborator sent me an email and he said the following thing, he told me, "Look, there are 3 different teams of physicists who have managed to stabilize the standard model until making it compatible with the mass of the Higgs.". Good then okay, I keep reading his email and he says "They did that by adding a scalar field which verifies certain coupling properties with vacuum.". Then, I continue to read his message, he said to me : "This scalar field, we had it in our article of 2010, but we neglected it.". So in fact, we had it. In fact, we were discouraged. What ? That is to say that we had said : "The scalar field, it changes nothing.". If we had been courageous, truly courageous, we would have taken it into account and we would have seen that everything worked wonderfully. So that's it, it's for that side.

And in fact, therefore, what happened if you will, in terms of my mathematical development is that after having developed the theory called non-commutative geometry, so it's a theory that is still... I shouldn't want to make you believe that this is a theory that is simple, it is a theory that is very elaborated, it has a lot of relationships with a lot of different concepts, etc. And basically, each of the concepts involved in the geometry we are used to has to be changed and we look at it in a completely different way.

Even the integral, even the notion of integral is changed, it is replaced by this which we call the Dixmier trace and which is a concept which was invented by Dixmier and which plays an absolutely central role in this theory. Okay, it's connected to a bunch of other theories, but I didn't mean to bother you with that. Now, I'm going to explain that to you...

So in fact, there was another rather bizarre phenomenon, which is that in the year 1996, I was invited back to Seattle. I couldn't refuse if I was invited back to Seattle, and it was for the birthday of Atle Selberg, who was a great number theorist, and I was invited because in my collaboration with Jean-Benoît Bost, we wrote an article in which we found that we had what's called a phase transition on a statistical mechanical system and we found that the partition function of the system was Riemann's zeta function.

So, as this lecture was a lecture on the Riemann zeta function, they invited me, so I did well, I followed the same route a bit than before. I stopped in Victoria, then after I went to Seattle and in Seattle, I gave my lecture and after the lecture, I saw Selberg who said to me : "It is not clear that what you are doing will be related to...". If you want of course, we know the famous conjecture. He told me that, he told me that. And well, of course, lying, I mean, we also like to be provoked, we like people to tell you critics. There's no shortage of that in mathematics, no problem on that.

And besides, I have to add one thing, it is that not only do people yell at you, but you have also good friends who pass the critics on to you. So it doesn't not lack, but it has a positive side. It has a positive side because not only you have to be catchy, but you have to be able to be, to transform frustration that you have when people criticize you in positive energy. This is absolutely crucial. It is an essential quality, that is to say if someone criticizes you on one way or another, you have to take it and consider it as an energy potential, not as something negative. You have to be able to distance yourself from yourself and see it as positive energy.

Okay, so when I went back from Seattle instead of... I didn't mind jet lag. That is to say ? I stayed on Seattle time, I stayed on the Seattle area and what I did was that for a week I didn't work.

I was reading the book called *The Staff*, I don't know if you know this book, it's a book about astronauts and Apollo 13, etc. On this whole story, it's a wonderful book. Well, I was reading this book, I read it, I tasted it on the spot, I could have read it in a few hours.

But in fact, I tasted it little by little and after a week, I understood that in fact, what people were looking for when they were looking for a spectral realization of zeta zeros, they were all looking for it in the form of what is called an emission spectrum, that is to say a spectrum in which you will have a black background and you will have emission lines. For example, if you are taking sodium and you heat it, the sodium will give you a spectrum. If you pass the light through a prism, it will give you a number of very bright lines but well isolated like that, on a black background. Okay, but actually it's not like that we saw the spectra for the first time. The spectra, it is Fraunhofer who really discovered them and he discovered them by taking the light that came from the sun, we had already looked at this light through a prism, the prism decomposes light in different colors, the colors of the rainbow.

But what Fraunhofer did, he had a great idea. He had the brilliant idea of looking at it under a microscope. And when he looked under the microscope, he realized that in fact, there were lots of black lines. First, he cleaned up his thing, okay. And then, in fact, he realized that whatever instrument he took, there were the same black stripes. So what ? The wonderful story that was behind is that afterwards, there are Bunsen and Kirchhoff who have succeeded in heating bodies like sodium, etc. produce the same spectrum, but to produce them as emission spectra. Not like black lines, the reverse, except that he still failed to try with the light of the sun. There were black lines that we couldn't produce by broadcast. So obviously physicists are always clever, they said these black lines correspond to a new body that they called Helium, like the sun, of course. When they have it called Helium, of course, it's a bit like dark matter.

You will tell me what is this story ? Except there was an eruption of Vesuvius. And when they did the emission spectrum of Vesuvius lava, they found helium in it. Okay, so what I understood when I got home from Seattle was that in fact, instead of looking for an emission spectrum, which was what people were looking for, but there was a "-" sign that didn't work, there was always a "-" sign in the formula that didn't work.

In fact, the spectrum had to be sought as an absorption spectrum. So what ? I already had the non-commutative space I needed for that. I already had the non-commutative space that was necessary. And I knew how to demonstrate that this non-commutative space gives good terms in what is called the Riemann formula. But I had seen so after, of course, as I knew we had to look for an absorption spectrum. I looked to see if that, indeed, gave the spectral realization. And then, the miracle is that it gave the spectral realization.

That's what I understood when I got back from Seattle. After a period of total boredom, that I cannot recommend highly enough. There were no emails at the time. Me, I was not connected. Okay. It was a good time to leave the brain function because I was reading something else. I was not doing math. Of course. I was reading something else and after a while it came as something which has become completely natural. So what did that mean ?

But that meant that this very abstract geometry that had come from afar, you see, who came from von Neumann, who came from Heisenberg, etc., it looked like it was working, it seemed to work as well for the space in which we live and for probably the most difficult space that exists, which is the one which understands the nature of prime numbers, of the set of prime numbers, because what is behind the spectral realization, the state function, etc., is exactly the nature of the set of prime numbers. So that was the starting point. I wrote a little note to the Accounts. I was invited to Princeton.

And then there, what happened, me, I call it a gag, I found it very funny. This means that I gave my conference at Princeton. Good, etc. It is true that what you have to know, is that the subject in question, which is Riemann's hypothesis as soon as we approache, it's a subject that is mined. So you should know that it is protected by mountains of skepticism. Okay. But there is always that I had found something anyway.

I went there. I made my presentation, I surely understood something, if you will, it was the understanding of the Riemann-Weil formula in the form of trace formula. And then there are two people that I knew and that I know very well. There is one person I had collaborated with, Paula Cohen and Bombieri who made a hoax. In fact, a hoax for April 1, that is, April 1, you know, we usually do hoaxes, we see stuff. So they sent a hoax by email saying that after my conference, there was a Russian who was there and who had succeeded in demonstrating Riemann's hypothesis. Okay, so it was very funny.

I laughed, etc. Except that I hadn't realized that, it was in 98, in March 98, and I didn't realize that this hoax was going to be sent everywhere. So it was sent pretty much everywhere. Inevitably, there were people who took it seriously. So what ? What is absolutely incredible is that it was taken seriously. It was in 1998. It's a year when the International Congress of Mathematicians was held in Berlin. Well, I have nothing against the Germans, but hey, if you want, there was, basically, there was Germans who took it seriously. So what did they decide to do ?

This is something that is absolutely incredible. I realized only recently because I mean, back then I had ignored this game. I shrugged. What did they decide to do ? They decided to invite for talk for an hour at convention, the person who was best suited to be my competitor. Okay. So instead of inviting me, for example, they invited a person who was working on the same subject and they gave him a boulevard.

And well, I remember that Selberg, when I saw him again the same year, it was after the congress. He told me that he had never heard such an empty presentation. And me, I didn't know at the time, I hadn't looked, I looked recently and I was flabbergasted because I realized that this presentation, in fact, is a presentation who used my ideas without really quoting them. Or rather by quoting them with what Grothendieck calls the technique thumb. The technique thumb, it's the following if that can serve to you, it's... you know, you're borrowing someone's idea. But you don't really want to quote him. You put his article in bibliography, but you quote him for something else. It works very, very well. Okay, so. Okay, so this is the prototype of what happened to me at that time. It's the prototype of the experiment which happens when we approach this subject, when we are interested in this subject, etc.

But on the other hand, you can't be afraid. You do not have to be afraid. That is an absolutely essential thing. You have to be able to endure this kind of slander without drawing any consequences. And precisely, trying to transform them into positive energy. Okay, so what has happened since then ? I will, I will not delay... I do not know. What has happened since ? If you want, what happened since that time, this is the next thing. Is that I collaborated with Katia Consani. And at the time, at the time of Grothendieck, I had only one idea, it was to flee the subjects that Grothendieck was dealing with. Why ? Because there was a kind of snobbery around. There was a kind of courtyard surrounding it, etc. It's for that I had done operator algebras. So there I started to learn the algebraic geometry with Katia Consani.

And in 2014, it was not long ago. It means that you should never discourage. In 2014, we made an absolutely incredible discovery. We have discovered that this non-commutative space that I had used to make the spectral realization, etc., but that people could think of as a completely weird because non-commutative, etc., in fact, it was the points space of a Grothendieck's topos of an incredible simplicity, which one calls the Topos of the frequencies, simply the half-line produced semi-direct by action, by integers, multiplication. So it's something that is wonderfully simple. But when we calculate the points of this topos, because the notion of Topos, it is sufficiently subtle so that when you calculate the points it's something. It is in general very, very difficult when you calculate the points of this topos. You find a non-commutative space and this non-commutative space is exactly the space that I built to have spectral realization.

And what did it give us with Katia Consani ? It gave us on this space the structural bundle, that is to say before, we would never have imagined that and we saw, we understood that this structural beam, it was in fact a beam which is called tropical, that is to say which is connected to what is called tropical geometry. And so, that allows us to move forward. It allows us to move forward. I did my last two lessons of the Collège largely on it.

So that allowed us, if you want, we are not far from the goal. Of course, you can't say it until you've got there, you can't say anything. One can say absolutely nothing.

What if we said something ? People would hit us with a hammer on our heads. Above all you have to say nothing, but if you want, what we discovered, it doesn't matter if you arrive or not. Because in this guess, what is wonderful, is that if we really know the underside of mathematics, we realize that this conjecture is at the root of most concepts that were developed during the XX$^{\text{th}}$ century.

I will not give you a description, but in fact, in almost every mathematical concepts that were developed, they were developed with that in mind, behind, so here, in fact, we came to a space. Now, it's also progressing : we're arrived at a space which is much more, how to say, geometric, which is beautiful and suddenly more understandable, but which is, how to say ? which is understandable because Grothendieck had this wonderful idea for topos. And this idea of Grothendieck's topos, in fact, this idea has the same relationship with non-commutative geometry as the relationship that exists in the Langlands program between the Galois theory and automorphic functions. So it's exactly the same, the same relationship that still appears.

So to finish, just one thing I wanted to say is that there is another collaboration that was crucial and that is what we managed to do with Chamseddine and Mukhanov who is a researcher who does cosmology, so what we managed to do : we managed to understand what was the deep root of the standard model coupled with gravitation. Because before, we put the fine structure I was talking about, we put it from experiences, we started from experimental results and everything that, it was said that it needed such algebra for it to stick with experience. And we had no conceptual reason to say why it was necessary to put this algebra and not an other one. And that, that reason, we found it and we found it by a contest of circumsctances. We were looking to solve a geometric problem, purely geometric, and by solving this geometric problem, we came across the good Clifford algebra that we had put at hand before.

So that's it. And there is a very deep theorem that shows that we made all the varieties like that. It is a theorem which, geometrically, is based on this kind of images. So that's it, I think I'm going to stop. I forgot to tell you something. Okay, but I forgot to tell you something, it is that in fact, what underlies my presentation is something that had already been understood by Shakespeare.

I'll tell you what Shakespeare writes, there's the translation too, but I tried to translate. Shakespeare is always much better than his translation. So, what Shakespeare says is this :

There is a tide in the affairs of men,
Which taken at the flood, leads on to fortune.
Omitted, all the voyage of their life

Is bound in shallows and in miseries.
On such a full sea are we now afloat.
And we must take the current when it serves
Or lose our ventures.

Il est une marée dans les affaires des hommes,
Qui, prise à son apogée, conduit à la fortune.
Ignorée, tout le voyage de leur vie
Est confiné aux bas-fonds et aux écueils.
Sur une telle mer, nous sommes maintenant à flots
Et devons suivre le courant quand il forcit
Ou réduire à néant nos projets.

It's the only thing I want you to remember : when you see the tide, you have to follow it. But you have to feel it, of course, you have to feel it is there, okay. It's intuition. This is called intuition, it is something that is impossible to define. It's not something you can rationalize, but it's something which is fundamental in the job we do.

((Applause).

One of the students of the ENS organizing the Maths for all Cycle : Thank you very much for this presentation. If you have any questions do not hesitate. We have a little time, I think.

Question : I believe Shakespeare said "The world is a theater.", is it the same for Mathematics ?

Yes, there is a lot of truth and it's great that you said that because that allows me to explain to you what a topos is. It's great, it's absolutely great, I'll explain what a topos is. Th common theater in mathematics, it is set theory. We all know the sets theory, we know the groups, we know the algebras, we know... the usual theater of mathematics. What is a topos ? It's something extraordinary because... the theater is the same, the actors are the same... But behind the scene, there is a kind of Deus Ex Machina, which makes things to vary, which introduces variability.

That means that in set theory, there will be variability and that means something extraordinary, which is that a geometric space, it is not perceived by what it is, it is not at the center of the stage at all. It's the Deus ex machina backstage who makes things happen. And it is by understanding how things vary that we understand the geometric space that is hidden behind all that. It is purely theater, it is purely theater.

Ordinary set theory is a static theater, but the topos is much more interesting. Okay, that's a fundamental idea of Grothendieck, but you, you will never see it explained like that in the books.

Question : This is a question that may have been asked a bit often, but can you discover mathematics, it's something that exists without rationality of man ?

Of course. So, of course, we discover... The reason... I had this very long discussion engaged with Changeux, with Jean-Pierre Changeux, I will say... Jean-Pierre Changeux wanted to demonstrate that in fact mathematics were a construction of the brain.

But no. But in my opinion, it does not hold. And the reason why it doesn't hold, it's the next thing, is that thanks to mathematics, we explain the table from Mendeleïev, we explain, if you like, why there are chemical bodies, etc., etc. Okay, so the comparison I always take, I take two comparisons. The comparison I take is, take Watson and Crick. When Watson and Crick discover the double helix structure of DNA, they discover it, they don't have invented it. They were not the ones who invented it. Math is exactly that, it's exactly the same, that is to say... And especially now when at the computer. If you want, it's terrible now, and that, I haven't talked about it. But having computers and computers that are so powerful, it makes it possible to collide to mathematical reality. But all the time, all the time. That is to say, whatever problem you have, you can always test it with the computer, always.

And if it is a problem of symbolic calculation, you can solve it with the computer. So, I'm going to say, the computer, it doesn't invent. I'll say shit, we tell it a problem, okay, I mean, it tells you if what you found is correct or not, etc. So no, it's a real reality, it's a real reality. It's not a reality which is concretely realized in the world as it is, but it's a reality that is just as resilient, just as impossible to change as the external reality.

That's for sure, for sure. We invent tools, because Watson and Crick observe the double helix. They use the electronic microscope. We invent tools, but there is a reality that is there, a reality that is there, that is impossible to change.

Question : Still in the same vein, do you think that we could also discover intuitions that were previously completely hidden from us, and that we end up doing math with things that come out of human power, recently discovered and that a new branch that we are exploring...

What kind of power recently discovered ?

Continuation of the student's question : That is, for example, when you are dealing with subjects like topos for example, these are not things that directly, we could grasp *by intuition which is not mathematically educated. Is, according to you, after a mastery of the field, more or less relative mastery, that we could approach an intuition...*

But then, this is a very good question. This is a very good question because the human mind is not trained in quantum. The human mind is trained to the classic and therefore the human spirit in particular used to always give a classic image of quantum things. What is certain is that there is more instruments now that are based on quantum. For example, there is an instrument that makes random numbers, which is made by a Swiss man and with a mobile phone, we make random numbers, we make them with quantum and quantum is more and more widespread now. So what ? If we actually managed to train in quantum, to train, well... it's obvious that quantum hasn't been used much for natural selection so far, so we are not trained for that. But if we did manage to get to know each other a lot more with quantum optics, with quantum, etc., it's absolutely obvious that we would make progress. It's obvious. This is obvious.

I'm not talking about machine learning and artificial intelligence because for me, it's exactly the opposite of what we do in math, that is to say that we seek to understand and we seek to invent, to invent tools, therefore to find the concepts behind what we discover. And that, well, machine learning solves problems. But if you solve a problem without knowing how, it's not really interesting.

Question : I have a question on this paper that you wrote with Mukhanov. I have got the impression that you are working with a riemannian metric but our world is not riemannian. So, does it work too ? or...

Of course it works. But the real answer is this, the real answer is that when we do that, physicists know this very well, is that when we do the field theory and when we do the functional integrals, etc., there is something that we call Feynmann's I_ε trick, that is to say that we add to the propagator a term I_ε, and if we think about what it means, it means that we are working in euclidian. So, in fact, we want to make the functional integrals in Euclidean and the true functional integral that we want to do, in this case, it is that we take two three-dimensional riemannian spaces and we look at the cobordisms between the two and we do the Euclidean integral on it, okay ? But it's perfectly true what you say.

Question : Do your mathematical theories allow you to understand what is quantum entanglement ?

Ah, it's... So there, I left again for an hour... So there, wonderful question, of course, but I have already made presentations which are on the Internet on this. Yes, if you want, that's a wonderful question. Why ? Because what I got you explained is that a non-commutative object generates its own time. So it is obvious that we want to understand in what sense, it is related to the time with which we are familiar, etc. We thought about it a lot and I had an episode that I tell, that we tell in our book *Le théâtre quantique* with a physicist named Carlo Rovelli.

And if you want what happens, it's the following. I thought a lot more, much more, afterwards, on this time that appears, etc. And what we said in the book on quantum theater, we have a sentence that sums up the idea. The sentence is "The quantum hazard is the ticking of the divine clock.", which means this is the quantum randomness, the fact that when we do an experiment twice, we pass an electron through a very small slit and we look at the place where it comes.

It will never arrive in the same place. We only know the probability, so that's the quantum hazard, and the theory that's based on it, if you will, is that precisely this quantum hazard generates time. But in fact, when we think about it further and because of the entanglement, we realize that we commit everytime a mistake. All the physics we know, it is written by equations as a function of time.

To go back to my year in Math Spé, I had a special math teacher and once, he had put me a curve on the board and he said "Mister Connes ? What is the parameter ?" *(AC draws a curve in the space in front of him.)* I thought, thought... Then, after a while, I said "It's time !". He was very happy. So you see, all the equations of physics, are written as dt, dt^2. This is that I actually think, to answer that, and I will answer your question.

I think that time is only an emerging variable and that true variability since we attribute all variability to the passage of time. But I think that true variability is more primitive than time and that this true variability, it is the quantum hazard. So what is the meaning of entanglement, the meaning entanglement is that the quantum hazard is synchronized in two events that are correlated. It is synchronized, that is to say instead of being completely random and purely independent, it is synchronized. So there should be a very deep reflection, which I am not able to have, and which would consist in saying that variability in physics, it comes from the quantum hazard and that time is just an emerging phenomenon and we should understand entanglement that way. Because the entanglement is incomprehensible, otherwise. Why is it misunderstood ? Because what says the entanglement of a phenomenon of quantum mechanics which is entangled, is that you're going to have, as soon as you make an observation on A, it's going to be reflected immediately on B. But it will not transmit information, but however, these are two events that are causally independent. So you cannot say that there is one that is before the other. So, it's completely incomprehensible. And what I'm saying is that precisely, the error, the terrible heresy, is to try to write everything over time, and that we need a deeper reflection and which should be based on these things.

This is a second question that one must ask you often too, but for you, is Riemann's hypothesis true ?

Well there, I can't resist the temptation. I can't resist the temptation, but I have no right to speak about it. It is that we are finishing a book whose we gave the preprints today to Odile Jacob and in which we tell a story which is the story of a mathematician, much like me, who has enough worked on this hypothesis and who is ready to sell his soul to the Devil. So he is ready to sell his soul to the Devil, not to recover youth or whatever else, because hey, he's sick of it. He is discouraged, he wants to know, he wants to sell his soul to the Devil. And then, if you like, the problem is how to meet the Devil. And in fact, one day he goes to a conference on machine learning. Then, he did a bunch of calculations, if you will, a bunch of stuff, and he recognizes in calculations that the guy does, because you know, we say in mathematics "the Devil is in the details", he recognizes in the talk of the guy, the machine learning specialist, he recognizes the Devil. So, I'm not going to tell you the rest of the story because you will know it in a book, I am not telling it to you in advance.

You don't want me to tell, Danye ?

So I'm not telling you, but you'll see that : it's a very, very elaborate story, and at the end of the book, well, there is a stuff like that, that's it. Now, there you go. So I'm going to say, like I said from the start, you cannot know anything until you are at the end and no doubt that we shouldn't get to the end, because there is another aspect of mathematics which I did not mention, because it is a more, how to say, more difficult aspect.

It's always the fear of being wrong and I imagine that if we got to the end, we wouldn't would live more because we would be constantly afraid of having made a mistake somewhere. And that would be an absolutely unbearable situation. Okay, this is not desirable, really. Okay, there you go. In any case, you will have all the details soon from the history, this history of the Devil.

One last question : Yes, it may be a bit prosaic, but just now, could you talk about computers ? You said you could solve symbolic calculations or that we could face reality. But yourself, you use computers or... ?

Terribly, terribly.

Right now, I'm plugged into the big machines at Polytechnique to do a calculation. I use it terribly.

Terribly, of course, of course. It's amazing. It's great because as soon as we have a little practice, we manage to put any problem. For example, the weirdest problem you can think of, you will think "no we can't have it resolved by a computer, yes, yes.". And we can understand a lot of things, a lot of things. Even for geometry problems, etc. So, because above all, the visualization, the ability to visualize, the *Manipulate* and all that.

It's amazing. It's amazing. It's a wonderful tool, wonderful. I cannot say enough that it is a wonderful tool.

Last question, yes.

Last question : I have a question. You seem to demonize the machine learning, there is something wrong with machine learning. So I would like to understand, precisely. Machine learning is yet another area, what have you tried to do with machine learning ? In fact ? and what dit you get upset that you were not able to do ?

Imagine that machine learning tells you "Riemann's hypothesis is true." but doesn't give you the reason, don't give you concepts that were invented for the occasion, etc. It would be sad, sad to die. So, what I blame, I don't blame, but I mean, I understood at one point, talking to Alain Prochiantz, the analogy that there was between natural selection and the machine learning. It is true that we arrive at a result, but if we arrive for example at a result, and we do not understand why and we do not develop a concept. I'm frustrated. Personally, I am very, very frustrated. If it's not renewable, if it can't be repeated.

You have to draw positive energy from it ;-)

What ? Ah yes, draw positive energy from it, there, I mean, there is something to do. Of course, of course. No, but I'm not saying, but for the moment, it doesn't work so great, because when you have a machine learning on the phone, it's really not terrible. "Repeat... I don't understand what you are saying...". Okay, it will get better, that's for sure.

The organizer : Thank you very much, again, for your presentation.

(Applause)