Invariant relations between binary Goldbach's decompositions'numbers coded in a 4 letters language

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October 2014

1 Introduction

Goldbach's conjecture states that each even integer except 2 is the sum of two prime numbers. This note presents a set of invariant relations uncovered between some Goldbach's decompositions'numbers, the decompositions being coded in a 4 letters language. We will try to use those invariant relations to attempt to obtain a recurrence demonstration of Goldbach's binary conjecture. In the following, one is interested in decompositions of an even number n as a sum of two odd integers $p+q$ with $3 \leqslant p \leqslant n/2$, $n/2 \leqslant q \leqslant n-3$ and $p \leqslant q$. We call p a *n's first range sommant* and q a *n's second range sommant*.

Notations :

We will designate by :

- a : an n's decomposition of the form $p+q$ with p and q primes ;
- b : an n's decomposition of the form $p+q$ with p compound and q prime ;
- c : an n's decomposition of the form $p+q$ with p prime and q compound ;
- d : an n's decomposition of the form $p+q$ with p and q compound numbers.

Example :

40	3	5	7	9	11	13	15	17	19
	37	35	33	31	29	27	25	23	21
l_{40}	a	c	c	b	a	c	d	a	c

2 The main array

We designate by $T = (L, C) = (\ l_{n,m}\ )$ the array containing $l_{n,m}$ elements that are one of a, b, c, d letters. n belongs to the set of even integers greater than or equal to 6. m, belonging to the set of odd integers greater than or equal to 3, is an element of list of n first range sommants.

Let us consider g function defined by :

$$\begin{array}{rccl} g: & 2\mathbb{N} & \rightarrow & 2\mathbb{N}+1 \\ & x & \mapsto & 2\left\lfloor \dfrac{x-2}{4} \right\rfloor + 1 \end{array}$$

$g(6) = 3, g(8) = 3, g(10) = 5, g(12) = 5, g(14) = 7, g(16) = 7, etc.$

$g(n)$ function defines the greatest of n first range sommants.

As we only consider n decompositions of the form $p+q$ where $p \leqslant q$, in T will only appear letters $l_{n,m}$ such that $m \leqslant 2\left\lfloor \frac{n-2}{4} \right\rfloor+1$ in such a way that T array first letters are : $l_{6,3}, l_{8,3}, l_{10,3}, l_{10,5}, l_{12,3}, l_{12,5}, l_{14,3}, l_{14,5}, l_{14,7}, etc.$

Here are first lines of array T.

C / L	3	5	7	9	11	13	15	17
6	a							
8	a							
10	a	a						
12	c	a						
14	a	c	a					
16	a	a	c					
18	c	a	a	d				
20	a	c	a	b				
22	a	a	c	b	a			
24	c	a	a	d	a			
26	a	c	a	b	c	a		
28	c	a	c	b	a	c		
30	c	c	a	d	a	a	d	
32	a	c	c	b	c	a	b	
34	a	a	c	d	a	c	b	a
36	c	a	a	d	c	a	d	a
...								

FIGURE 1 : words of even numbers between 6 and 36

Remarks :

1) words on array's diagonals called *diagonal words* have their letters either in $A_{ab} = \{a, b\}$ alphabet or in $A_{cd} = \{c, d\}$ alphabet.

2) a diagonal word codes decompositions that have the same second range sommant.
For instance, on Figure 4, letters *aaabaa* of the diagonal that begins at letter $l_{26,3} = a$ code decompositions $3 + 23, 5 + 23, 7 + 23, 9 + 23, 11 + 23$ and $13 + 23$.

3) let us designate by l_n the line whose elements are $l_{n,m}$. Line l_n contains $\left\lfloor \frac{n-2}{4} \right\rfloor$ elements.

4) n being fixed, let us call $C_{n,3}$ the column formed by $l_{k,3}$ for $6 \leqslant k \leqslant n$.

In this column $C_{n,3}$, let us distinguish two parts, the “top part” and the “bottom part” of the column.

Let us call $H_{n,3}$ column's “top part”, i.e. set of $l_{k,3}$ where $6 \leqslant k \leqslant \left\lfloor \frac{n+4}{2} \right\rfloor$.

Let us call $B_{n,3}$ column's “bottom part”, i.e. set of $l_{k,3}$ where $\left\lfloor \frac{n+4}{2} \right\rfloor < k \leqslant n$.

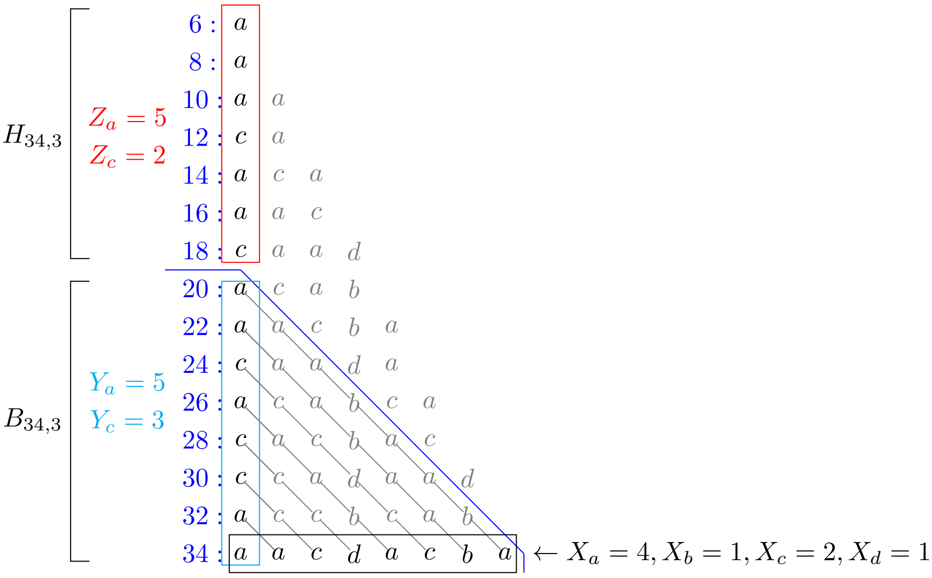


FIGURE 2 : $n = 34$

To better understand countings in next section, we will use projection P of line n on bottom part of first column $B_{n,3}$ that "associates" letters at both extremities of a diagonal (it associates to the letter that codes $p+q$ decomposition the letter that codes $3+q$ decomposition). If we consider application *proj* such that $proj(a) = proj(b) = a$ and $proj(c) = proj(d) = c$ then, since 3 is prime, $proj(l_{n,2k+1}) = l_{n-2k+2,3}$.

We can also understand the effect of this projection (that preserves second range sommant) by analyzing decompositions :

- if $p+q$ is coded by an a or a b letter, it corresponds to two possible cases in which q is prime, and so $3+q$ decomposition, containing two prime numbers, will be coded by an a letter ;
- if $p+q$ is coded by a c or a d letter, it corresponds to two possible cases in which q is compound, and so $3+q$ decomposition, of the form $prime + compound$ will be coded by a c letter.

We will also use in next section projection P' of line n on top part of the first column $H_{n,3}$ that associates to the letter that codes $p+q$ decomposition the letter that codes $3+p$ decomposition (let us note that *first range sommant* becomes *second range sommant*) ; let us analyze how such a projection will affect decompositions :

- if $p+q$ is coded by an a or c letter, it corresponds to the two possible cases in which p is prime, then $3+p$ decomposition, containing two prime numbers will be coded by an a letter ;
- if $p+q$ is coded by a b or d letter, it corresponds to the two possibl cases in which p is compound, then $3+p$ decomposition, of the form $prime + compound$ will be coded by a c letter.

3 Computations

1) We note :

- $X_a(n)$ the number of n's decompositions of the form $prime + prime$;
- $X_b(n)$ the number of n's decompositions of the form $compound + prime$;
- $X_c(n)$ the number of n's decompositions of the form $prime + compound$;
- $X_d(n)$ the number of n's decompositions of the form $compound + compound$.

$X_a(n), X_b(n), X_c(n), X_d(n)$ variables are counting logical assertions numbers.

$X_a(n) + X_b(n) + X_c(n) + X_d(n) = \left\lfloor \dfrac{n-2}{4} \right\rfloor$ is the number of elements of line n.

Example : $n = 34$:
$X_a(34) = \#\{3+31, 5+29, 11+23, 17+17\} = 4$
$X_b(34) = \#\{15+19\} = 1.$
$X_c(34) = \#\{7+27, 13+21\} = 2$
$X_d(34) = \#\{9+25\} = 1$

2) Let $Y_a(n)$ (resp. $Y_c(n)$) being the number of a letters (resp. c) that appear in $B_{n,3}$. We recall that there are only a and c letters in first column because it contains letters associated with decompositions of the form $3+x$ and because 3 is prime.

Example :

- $Y_a(34) = \#\{3+17, 3+19, 3+23, 3+29, 3+31\} = 5$
- $Y_c(34) = \#\{3+21, 3+25, 3+27\} = 3$

3) Because of P projection that is a bijection, and because of a, b, c, d letters definitions, $Y_a(n) = X_a(n) + X_b(n)$ and $Y_c(n) = X_c(n) + X_d(n)$. Thus, trivially, $Y_a(n) + Y_c(n) = X_a(n) + X_b(n) + X_c(n) + X_d(n) = \left\lfloor \dfrac{n-2}{4} \right\rfloor$.

Example :

$$\begin{aligned}
Y_a(34) &= \#\{3+17, 3+19, 3+23, 3+29, 3+31\} \\
X_a(34) &= \#\{3+31, 5+29, 11+23, 17+17\} \\
X_b(34) &= \#\{15+19\} \\
\\
Y_c(34) &= \#\{3+21, 3+25, 3+27\} \\
X_c(34) &= \#\{7+27, 13+21\} \\
X_d(34) &= \#\{9+25\}
\end{aligned}$$

4) Let $Z_a(n)$ (resp. $Z_c(n)$) being the number of a letters (resp. c) that appear in $H_{n,3}$.

Example :

- $Z_a(34) = \#\{3+3, 3+5, 3+7, 3+11, 3+13\} = 5$
- $Z_c(34) = \#\{3+9, 3+15\} = 2$

$Z_a(n) + Z_c(n) = \left\lfloor \dfrac{n-4}{4} \right\rfloor$.

Reminding identified properties

$$Y_a(n) = X_a(n) + X_b(n) \tag{1}$$

$$Y_c(n) = X_c(n) + X_d(n) \tag{2}$$

$$Y_a(n) + Y_c(n) = X_a(n) + X_b(n) + X_c(n) + X_d(n) = \left\lfloor \frac{n-2}{4} \right\rfloor \tag{3}$$

$$Z_a(n) + Z_c(n) = \left\lfloor \frac{n-4}{4} \right\rfloor \tag{4}$$

Let us add four new properties to those ones :

$$X_a(n) + X_c(n) = Z_a(n) + \delta_{2p}(n) \tag{5}$$

with $\delta_{2p}(n)$ equal to 1 when n is the double of a prime number and equal to 0 otherwise.

$$X_b(n) + X_d(n) = Z_c(n) + \delta_{c-imp}(n) \tag{6}$$

with $\delta_{2odd-compound}(n)$ equal to 1 when n is the double of a compound odd number and equal to 0 otherwise (when there exists k such that $n = 4k$ (doubles of even numbers) or when n is the double of a prime number).

$$Z_c(n) - Y_a(n) = Y_c(n) - Z_a(n) - \delta_{4k+2}(n) \tag{7}$$

with $\delta_{4k+2}(n)$ equal to 1 when n is the double of an odd number and 0 otherwise.

$$Z_c(n) - Y_a(n) = X_d(n) - X_a(n) - \delta_{2odd-compound}(n) \tag{8}$$

Properties 1, 2 and 3 come simply from projection P's definition.

Properties 5 and 6 come simply from projection P''s definition.

One can notice a certain redundancy between the 3 booleans that are introduced here $\delta_{2p}(n), \delta_{2odd-compound}(n)$ and $\delta_{4k+2}(n)$. Trivial logical assertion $(\delta_{2p}(n) \vee \delta_{2odd-compound}(n)) \rightarrow \delta_{4k+2}(n)$ is always verified.

4 Demonstrations

4.1 Utilitaries

Let us demonstrate that if n is an odd number double (i.e. of the form $4k+2$), then $\left\lfloor \frac{n-2}{4} \right\rfloor = \left\lfloor \frac{n-4}{4} \right\rfloor + 1$.
Indeed, the left part of the equality is equal to $\left\lfloor \frac{(4k+2)-2}{4} \right\rfloor = \left\lfloor \frac{4k}{4} \right\rfloor = k$.
The right part of the equality is equal to $\left\lfloor \frac{(4k+2)-4}{4} \right\rfloor + 1 = \left\lfloor \frac{4k-2}{4} \right\rfloor + 1 = (k-1) + 1 = k$.

Let us demonstrate that if n is an even number double (i.e. of the $4k$), then $\left\lfloor \frac{n-2}{4} \right\rfloor = \left\lfloor \frac{n-4}{4} \right\rfloor$.
$\left\lfloor \frac{4k-2}{4} \right\rfloor = k-1$ and $\left\lfloor \frac{4k-4}{4} \right\rfloor = k-1$.

4.2 Properties 7 and 8

4.2.1 Property 7

Property 7 enonciates that $Z_c(n) - Y_a(n) = X_d(n) - X_a(n) - \delta_{2odd-compound}(n)$ with $\delta_{2odd-compound}(n)$ that equals 1 if n is the double of an odd compound integer and equals 0 otherwise.

By definition, $Z_c(n)$ counts the number of decompositions of the form $\alpha = 3 + compound$ with $compound$ strictly lesser than $n/2$ (let us call E this decompositions'set). So $Z_c(n)$ counts also the number of n's decompositions of the form $\beta = compound + y$ by bijection of second range sommant of α decomposition on first range sommant of β decomposition. But the number of decompositions of the form $compound + y$ is equal to $X_b(n) + X_d(n)$ by definition of those variables.

By definition, $Y_a(n)$ counts the number of decompositions of the form $\gamma = 3 + prime$ with $prime$ strictly greater than $n/2$ (let us call F this decompositions'set). So $Y_a(n)$ counts also the number of n's decompositions of the form $\eta = x + prime$ by bijection of second range sommant of γ decomposition on second range sommant of η decomposition. But the number of decompositions of the form $x + prime$ is equal to $X_b(n) + X_a(n)$ by definition of those variables.

n's decompositions of the form $compound + prime$ are at the same time in E and in F. By computing $Z_c(n) - Y_a(n)$, we also obtain the value of $X_d(n) - X_a(n)$ by definition of what variables $Y_a(n), Z_c(n), X_d(n)$ et $X_a(n)$ count.

4.2.2 Property 8

Let us demonstrate that $Z_c(n) - Y_a(n) = Y_c(n) - Z_a(n) - \delta_{4k+2}(n)$ with $\delta_{4k+2}(n)$ that equals 1 if n is the double of an odd number ($\exists k \geqslant 3, n = 4k+2$) and 0 otherwise.

We use a recurrence reasoning :

i) We initialize recurrences according to the 3 types of numbers that must be studied : even doubles de pairs ($4k$), odd doubles ($4k+2$) with the odd being prime or compound.

Proprety 8 is true for $n = 14, 16$ and 18. Let us provide variables values in an array for those 3 even numbers :

n	$Z_c(n)$	$Y_a(n)$	$Y_c(n)$	$Z_a(n)$	δ_{4k+2}
14	0	2	1	2	1
16	0	2	1	3	0
18	0	2	2	3	1

ii) Property 8 is equivalent to $Z_a(n) + Z_c(n) + \delta_{4k+2}(n) = Y_a(n) + Y_c(n)$.
Four cases must be considered : two cases wherein n is an odd's double (prime or compound) and $n+2$ is an even's double and two cases wherein n is an even's double and $n+2$ is an odd's double (prime or compound).

iia) *n even's double and $n+2$ prime's double* :

n	δ_{2p}	$\delta_{2odd-compound}$	δ_{4k+2}
n	0	0	0
$n+2$	1	0	1

We make the hypothesis that property 8 is true for n,

$$Z_a(n) + Z_c(n) + \delta_{4k+2}(n) = Y_a(n) + Y_c(n) \quad (H)$$

Let us demonstrate that it is true for $n+2$,

$$Z_a(n+2) + Z_c(n+2) + \delta_{4k+2}(n+2) = Y_a(n+2) + Y_c(n+2) \quad (Ccl)$$

We have $Z_a(n+2) = Z_a(n)$ and $Z_c(n+2) = Z_c(n)$.

Let us remain property 3 concerning Ys:

$$Y_a(n) + Y_c(n) = \left\lfloor \frac{n-2}{4} \right\rfloor \quad (3)$$

In (Ccl), we can, par by our recurrence hypothesis and by property (3), replace the left member of the equality by $Z_a(n) + Z_c(n) + 1$ and then by $Y_a(n) + Y_c(n) + 1$ (by (H)) and then by $\left\lfloor \frac{n-2}{4} \right\rfloor + 1$ (by (3)) that is equal to $\left\lfloor \frac{n}{4} \right\rfloor$.

But in (Ccl), we can also replace the right member of the equality by $\left\lfloor \frac{n}{4} \right\rfloor$ because of property (3).

We have for $n+2$ an equality between left and right sides, i.e. property 8 is verified by $n+2$. From the hypothesis that property is true for n, we have inferred property is true for $n+2$.

iib) *n even's double and $n+2$ odd compound's double* :

n	δ_{2p}	$\delta_{2odd-compound}$	δ_{4k+2}
n	0	0	0
$n+2$	0	1	1

Let us make the hypothesis that property 8 is true for n,

$$Z_a(n) + Z_c(n) + \delta_{4k+2}(n) = Y_a(n) + Y_c(n) \quad (H)$$

Let us demonstrate it is true for $n+2$,

$$Z_a(n+2) + Z_c(n+2) + \delta_{4k+2}(n+2) = Y_a(n+2) + Y_c(n+2) \quad (Ccl)$$

We have $Z_a(n+2) = Z_a(n)$ and $Z_c(n+2) = Z_c(n)$.

We have $Y_a(n+2) = Y_a(n) + 1$ and $Y_c(n+2) = Y_c(n)$.

Let us recall property 3 concerning Ys :

$$Y_a(n) + Y_c(n) = \left\lfloor \frac{n-2}{4} \right\rfloor \quad (3)$$

In (Ccl), one can, by recurrence hypothesis and by property 3, replace equality's left member by $Z_a(n) + Z_c(n) + 1$, then by $Y_a(n) + Y_c(n) + 1$ (because of (H)), and then by $\left\lfloor \frac{n-2}{4} \right\rfloor + 1$ (because of (3)) that is equal to $\left\lfloor \frac{n}{4} \right\rfloor$.

But in (Ccl), one can also replace equality's right member by $\left\lfloor \frac{n}{4} \right\rfloor$ because of $Y_a(n)$'s and $Y_c(n)$'s evolutions.

There is also for $n+2$ equality between left and right members of the equation, i.e. property 8 is verified by $n+2$. From the hypothesis that property is true for n, we deduced property is true for $n+2$.

iic) *n prime's double and $n+2$ even's double* :

n	δ_{2p}	$\delta_{2odd-compound}$	δ_{4k+2}
n	1	0	1
$n+2$	0	0	0

Let us make the hypothesis that property 8 is true for n,

$$Z_a(n) + Z_c(n) + \delta_{4k+2}(n) = Y_a(n) + Y_c(n) \quad (H)$$

Let us demonstrate it is true for $n+2$,

$$Z_a(n+2) + Z_c(n+2) + \delta_{4k+2}(n+2) = Y_a(n+2) + Y_c(n+2) \quad (Ccl)$$

We have $Z_a(n+2) = Z_a(n) + 1$ et $Z_c(n+2) = Z_c(n)$.

Let us recall property 3 concerning Ys :

$$Y_a(n) + Y_c(n) = \left\lfloor \frac{n-2}{4} \right\rfloor \quad (3)$$

In (Ccl), one can, by recurrence hypothesis and by property 3, replace equality's left member by $Z_a(n) + Z_c(n)+1$ then by $Y_a(n)+Y_c(n)$ (because of (H)), and then by $\left\lfloor \frac{n-2}{4} \right\rfloor$ (because of (3)) that is equal to $\left\lfloor \frac{n}{4} \right\rfloor$.

But in (Ccl), one can also replace equality's right member by $\left\lfloor \frac{n}{4} \right\rfloor$ because of property (3).

There is also for $n+2$ equality between left and right members of the equation, i.e. property 8 is verified by $n+2$. From the hypothesis that property is true for n, we deduced property is true for $n+2$.

iid) *n odd compound's double and $n+2$ even double* :

n	δ_{2p}	$\delta_{2odd-compound}$	δ_{4k+2}
n	0	1	1
$n+2$	0	0	0

Let us make the hypothesis that property 8 is true for n,

$$Z_a(n) + Z_c(n) + \delta_{4k+2}(n) = Y_a(n) + Y_c(n) \qquad (H)$$

Let us demonstrate it is true for $n+2$,

$$Z_a(n+2) + Z_c(n+2) + \delta_{4k+2}(n+2) = Y_a(n+2) + Y_c(n+2) \qquad (Ccl)$$

We have $Z_a(n+2) = Z_a(n)$ et $Z_c(n+2) = Z_c(n) + 1$.

Let us recall property 3 concerning Ys :

$$Y_a(n) + Y_c(n) = \left\lfloor \frac{n-2}{4} \right\rfloor \qquad (3)$$

In (Ccl), one can, by recurrence hypothesis and by property 3, replace equality's left member by $Z_a(n) + Z_c(n) + 1$ puis par $Y_a(n) + Y_c(n)$ (because of (H)) then by $\left\lfloor \frac{n-2}{4} \right\rfloor$ (because of (3)) that is equal to $\left\lfloor \frac{n}{4} \right\rfloor$.

But in (Ccl), one can also replace equality's right member by $\left\lfloor \frac{n}{4} \right\rfloor$ because of property (3).

There is also for $n+2$ equality between left and right members of the equation, i.e. property 8 is verified by $n+2$. From the hypothesis that property is true for n, we deduced property is true for $n+2$.

5 Variables evolution

5.1 Counting decompositions of the forme $3+p$ (called "bottom-ones")

1) $Z_a(n+2) = Z_a(n)$ and $Z_c(n+2) = Z_c(n)$ when n is an even double. Indeed, $Z_a(n)$ (resp. $Z_c(n)$) counts number of primes (resp. odd compound numbers) strictly lesser than $\frac{n}{2}$ and there are as many as primes (resp. odd compound numbers) lesser than an even number than there are primes (resp. odd compound numbers) lesser than its immediate successor.

2) $Z_a(n+2) = Z_a(n) + 1$ and $Z_c(n+2) = Z_c(n)$ when n is a prime double ;

3) $Z_a(n+2) = Z_a(n)$ and $Z_c(n+2) = Z_c(n) + 1$ when n is an odd compound number's double.

5.2 Counting decompositions of the form $3+q$ (called "top-ones")

In the case in which $n+2$ is an odd's double, we only add a number to $H_{n+2,3}$ interval ; if this number $n-1$ is prime (resp. compound), $Y_a(n+2) = Y_a(n) + 1$ (resp. $Y_c(n+2) = Y_c(n) + 1$).

In the case in which $n+2$ is an even's double, we only add a number to the top extremity of the interval and we take off a number at the bottom extremity of the interval. From this fact, 4 cases have to be studied. Let us study how the decompositions'set $H_{n+2,3}$ evoluate.

- if $n-1$ and $\frac{n}{2}$ are both primes, we take off at the bottom and we put on at the top of the interval $H_{n+2,3}$ two letters of same kind, so $Y_a(n+2) = Y_a(n)$;
- if $n-1$ is prime and $\frac{n}{2}$ is compound then $Y_a(n+2) = Y_a(n) + 1$;

- if $n-1$ is compound and $\frac{n}{2}$ is prime then $Y_a(n+2) = Y_a(n) - 1$;
- if $n-1$ and $\frac{n}{2}$ are both compound, we take off at the bottom and we put on at the top of the interval $H_{n+2,3}$ two letters of same kind, then $Y_a(n+2) = Y_a(n)$.

6 Using variables'gaps

We are going to show that $X_a(n)$ can never be equal to 0 for $n \geqslant C$, i.e. that every even number $n \geqslant C$ can be written as a sum of two primes, i.e. verifies binary Goldbach's conjecture.

Let us present in more details what variables $Z_a(n), Z_c(n), Y_a(n)$ et $Y_c(n)$ represent.

- $Z_a(n)$ counts with a maximum error of 2 the number of prime numbers that are lesser than or equal to $\frac{n}{2}$;

$$Z_a(n) = \pi\left(\frac{n}{2}\right) + \delta_{Z_a}(n) \tag{9}$$

  with $\delta_{Z_a}(n)$ equal to -2 if n is a prime's double and equal to -1 otherwise ;

- $Z_c(n)$ counts with a maximum error of 1 the number of odd compound numbers that are lesser than or equal to $\frac{n}{2}$;

$$Z_c(n) = \left\lfloor \frac{n}{4} \right\rfloor - \pi\left(\frac{n}{2}\right) + \delta_{Z_c}(n) \tag{10}$$

  with $\delta_{Z_c}(n)$ equal to 1 if n is a prime's double and equal to 0 otherwise ;

- $Y_a(n)$ count with a maximum error of 1 the number of prime numbers that are between $\frac{n}{2}$ and n ;

$$Y_a(n) = \pi(n) - \pi\left(\frac{n}{2}\right) + \delta_{Y_a}(n) \tag{11}$$

  with $\delta_{Y_a}(n)$ equal to $-1, 0$ or 1 ($\delta_{Y_a}(n)$ is equal to 0 when $n-1$ and $n/2$ are both primes or both compound, $\delta_{Y_a}(n)$ is equal to -1 when $n-1$ is prime while $n/2$ is compound, and at last, $\delta_{Y_a}(n)$ is equal to 1 when $n-1$ is compound while $n/2$ is prime) ;

- $Y_c(n)$ counts with a maximum error of 1 the number of odd compound numbers that are between $\frac{n}{2}$ and n ;

$$Y_c(n) = \left\lfloor \frac{n}{4} \right\rfloor - \pi(n) + \pi\left(\frac{n}{2}\right) + \delta_{Y_c}(n) \tag{12}$$

  with $\delta_{Y_c}(n)$ equal to $-1, 0$ or 1. $\delta_{Y_c}(n)$'s values will be provided in a detailled way in section demonstrating property 12.

6.1 Demonstrations of properties 9, 10, 11 and 12

6.1.1 Property 9 : invariant relation concerning $Z_a(n)$

i) Initialisation : (9) is true for $n = 14, 16,$ 18 and 20.

Indeed, we have :

n	$Z_a(n)$	$\pi\left(\frac{n}{2}\right)$	$\delta_{Z_a}(n)$
14	2	4	-2
16	3	4	-1
18	3	4	-1
20	3	4	-1
22	3	5	-2

ii) Let us demonstrate by recurrence that if (9) is true for n, it is also true for $n+2$.

Let us make the hypothesis that property 9 is true for n,

$$Z_a(n) = \pi\left(\frac{n}{2}\right) + \delta_{Z_a}(n) \qquad (H)$$

Let us demonstrate it is true for $n+2$,

$$Z_a(n+2) = \pi\left(\frac{n+2}{2}\right) + \delta_{Z_a}(n+2) \qquad (Ccl)$$

Let us distinguish 4 cases, as in paragraph 4.2.2.

iia) n even's double, $n+2$ prime's double.

In that case, $Z_a(n+2) = Z_a(n)$, $\pi\left(\frac{n+2}{2}\right) = \pi\left(\frac{n}{2}\right) + 1$ and $\delta_{Z_a}(n) = -1$.

So we have,

$$\begin{aligned} Z_a(n+2) &= Z_a(n) \\ &= \pi\left(\frac{n}{2}\right) + \delta_{Z_a}(n) \\ &= \pi\left(\frac{n+2}{2}\right) - 1 + \delta_{Z_a}(n) \\ &= \pi\left(\frac{n+2}{2}\right) - 1 - 1 \\ &= \pi\left(\frac{n+2}{2}\right) + \delta_{Z_a}(n+2) \end{aligned}$$

with $\delta_{Z_a}(n+2) = -2$ as expected since in that case, $n+2$ is a prime's double.

iib) n even's double, $n+2$ odd compound's double.

In that case, $Z_a(n+2) = Z_a(n)$, $\pi\left(\frac{n+2}{2}\right) = \pi\left(\frac{n}{2}\right)$ and $\delta_{Z_a}(n) = -1$.

So we have,

$$\begin{aligned} Z_a(n+2) &= Z_a(n) \\ &= \pi\left(\frac{n}{2}\right) + \delta_{Z_a}(n) \\ &= \pi\left(\frac{n+2}{2}\right) - 1 \\ &= \pi\left(\frac{n+2}{2}\right) + \delta_{Z_a}(n+2) \end{aligned}$$

with $\delta_{Z_a}(n+2) = -1$ as expected since in that case, $n+2$ is an odd compound's double.

iic) n prime's double, $n+2$ even's double.

In that case, $Z_a(n+2) = Z_a(n) + 1$, $\pi\left(\frac{n+2}{2}\right) = \pi\left(\frac{n}{2}\right)$ and $\delta_{Z_a}(n) = -2$.

So we have,

$$\begin{aligned} Z_a(n+2) &= Z_a(n) + 1 \\ &= \pi\left(\frac{n}{2}\right) + \delta_{Z_a}(n) + 1 \\ &= \pi\left(\frac{n+2}{2}\right) - 2 + 1 \\ &= \pi\left(\frac{n+2}{2}\right) - 1 \\ &= \pi\left(\frac{n+2}{2}\right) + \delta_{Z_a}(n+2) \end{aligned}$$

with $\delta_{Z_a}(n+2) = -1$ as expected since in that case, $n+2$ is an even's double.

iid) n odd compound's double, $n+2$ even's double.

In that case, $Z_a(n+2) = Z_a(n)$, $\pi\left(\dfrac{n+2}{2}\right) = \pi\left(\dfrac{n}{2}\right)$ and $\delta_{Z_a}(n) = -1$.

So we have,

$$\begin{aligned} Z_a(n+2) &= Z_a(n) \\ &= \pi\left(\frac{n}{2}\right) + \delta_{Z_a}(n) \\ &= \pi\left(\frac{n+2}{2}\right) - 1 \\ &= \pi\left(\frac{n+2}{2}\right) + \delta_{Z_a}(n+2) \end{aligned}$$

with $\delta_{Z_a}(n+2) = -1$ as expected since in that case, $n+2$ is an even's double.

6.1.2 Property 10 : invariant relation concerning $Z_c(n)$

i) Initialisation : (10) is true for $n = 14, 16, 18$ and 20.

We have :

n	$Z_c(n)$	$\left\lfloor \frac{n}{4} \right\rfloor$	$\pi\left(\frac{n}{2}\right)$	$\delta_{Z_c}(n)$
14	0	3	4	1
16	0	4	4	0
18	0	4	4	0
20	1	5	4	0
22	1	5	5	1

and we verify that (10) is true for those 4 numbers.

ii) Let us demonstrate by recurrence that if (10) is true for n, it is also true for $n+2$.

Let us make the hypothesis that property 10 is true for n,

$$Z_c(n) = \left\lfloor \frac{n}{4} \right\rfloor - \pi\left(\frac{n}{2}\right) + \delta_{Z_c}(n) \qquad (H)$$

Let us demonstrate it is also true for $n+2$,

$$Z_c(n+2) = \left\lfloor \frac{n+2}{4} \right\rfloor - \pi\left(\frac{n+2}{2}\right) + \delta_{Z_c}(n+2) \qquad (Ccl)$$

Let us study our 4 cases again.

iia) n even's double, $n+2$ prime's double.

Dans ce cas, $Z_c(n+2) = Z_c(n)$, $\left\lfloor \dfrac{n+2}{4} \right\rfloor = \left\lfloor \dfrac{n}{4} \right\rfloor$, $\pi\left(\dfrac{n+2}{2}\right) = \pi\left(\dfrac{n}{2}\right) + 1$ et $\delta_{Z_c}(n) = 0$.

In that case,

$$\begin{aligned} Z_c(n+2) &= Z_c(n) \\ &= \left\lfloor \frac{n}{4} \right\rfloor - \pi\left(\frac{n}{2}\right) + \delta_{Z_c}(n) \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi\left(\frac{n+2}{2}\right) + 1 + \delta_{Z_c}(n) \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi\left(\frac{n+2}{2}\right) + 1 + 0 \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi\left(\frac{n+2}{2}\right) + \delta_{Z_c}(n+2) \end{aligned}$$

with $\delta_{Z_c}(n+2) = 1$ as expected since in that case $n+2$ is a prime's double.

iib) n even's double, $n+2$ odd compound's double.

In that case, $Z_c(n+2) = Z_c(n)$, $\left\lfloor \dfrac{n+2}{4} \right\rfloor = \left\lfloor \dfrac{n}{4} \right\rfloor$, $\pi\left(\dfrac{n+2}{2}\right) = \pi\left(\dfrac{n}{2}\right)$ et $\delta_{Z_c}(n) = 0$.

So we have,

$$\begin{aligned} Z_c(n+2) &= Z_c(n) \\ &= \left\lfloor \frac{n}{4} \right\rfloor - \pi\left(\frac{n}{2}\right) + \delta_{Z_c}(n) \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi\left(\frac{n+2}{2}\right) + \delta_{Z_c}(n) \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi\left(\frac{n+2}{2}\right) + 0 \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi\left(\frac{n+2}{2}\right) + \delta_{Z_c}(n+2) \end{aligned}$$

with $\delta_{Z_c}(n+2) = 0$ as expected in that case since $n+2$ is an odd compound's double.

iic) n prime's double, $n+2$ even's double.

In that case, $Z_c(n+2) = Z_c(n)$, $\left\lfloor \frac{n+2}{4} \right\rfloor = \left\lfloor \frac{n}{4} \right\rfloor + 1$, $\pi\left(\frac{n+2}{4}\right) = \pi\left(\frac{n}{2}\right)$ et $\delta_{Z_c}(n) = 1$.

So we have,

$$\begin{aligned} Z_c(n+2) &= Z_c(n) \\ &= \left\lfloor \frac{n}{4} \right\rfloor - \pi\left(\frac{n}{2}\right) + \delta_{Z_c}(n) \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi\left(\frac{n+2}{4}\right) - 1 + \delta_{Z_c}(n) \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi\left(\frac{n+2}{2}\right) - 1 + 1 \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi\left(\frac{n+2}{2}\right) + \delta_{Z_c}(n+2) \end{aligned}$$

with $\delta_{Z_c}(n+2) = 0$ as expected since in that case $n+2$ is an even's double.

iid) n odd compound's double, $n+2$ even's double.

In that case, $Z_c(n+2) = Z_c(n) + 1$, $\left\lfloor \frac{n+2}{4} \right\rfloor = \left\lfloor \frac{n}{4} \right\rfloor + 1$, $\pi\left(\frac{n+2}{2}\right) = \pi\left(\frac{n}{2}\right)$ and $\delta_{Z_c}(n) = 0$.

So we have,

$$\begin{aligned} Z_c(n+2) &= Z_c(n) \\ &= \left\lfloor \frac{n}{4} \right\rfloor - \pi\left(\frac{n}{2}\right) + \delta_{Z_c}(n) \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - 1 - \pi\left(\frac{n+2}{2}\right) + 1 + \delta_{Z_c}(n) \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi\left(\frac{n+2}{2}\right) - 1 + 1 \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi\left(\frac{n+2}{2}\right) + \delta_{Z_c}(n+2) \end{aligned}$$

with $\delta_{Z_c}(n+2) = 0$ as expected since in that case $n+2$ is an even's double.

6.1.3 Property 11 : invariant relation concerning $Y_a(n)$

The objective is to demonstrate why the invariant relation concerning $Y_a(n)$ and that is :

$$Y_a(n) = \pi(n) - \pi\left(\frac{n}{2}\right) + \delta_{Y_a}(n)$$

is always verified. In this aim, 16 cases are to be distinguished, according to primality characters of 4 numbers : $\frac{n}{2}$, $\frac{n+2}{2}$, $n-1$ and $n+1$, those primality characters having an influence on $Y_a(n)$, $\pi(n)$, $\pi\left(\frac{n}{2}\right)$ and $\delta_{Y_a}(n)$ evolutions.

i) recurrences initialisations : for 5 cases among the 16 to be envisagd, there is a contradiction (we called them $C1$ to $C5$ in bottom of the array below, contradiction coming for cases $C1$ to $C4$ from the fact that one can't have simmultaneously $n/2$ and $(n+2)/2$ that are both prime since they are consecutive integers) ; $C5$ is a contradiction because if $n-1$ and $n+1$ ar twin primes, their "father" (the even number between them) can be divided by 3 and so the half of this father, that is $n/2$, can't be a prime number too ; this is represented by crosses in first, third and fourth columns). For the remaining 11 cases, we must initialize recurrences. In the array below, we provide values that permit easily to verify that for small integers, property 11 is systematically verified.

n	$n/2$ *is prime*	$(n+2)/2$ *is prime*	$n-1$ *is prime*	$n+1$ *is prime*	$Y_a(n)$	$\pi(n)$	$\pi(n/2)$	$\delta_{Y_a}(n)$
14	×	−	×	−	2	6	4	0
16	−	−	−	×	2	6	4	0
18	−	−	×	×	2	7	4	−1
20	−	×	×	−	3	8	4	−1
22	×	−	−	×	4	8	5	1
26	×	−	−	−	4	9	6	1
36	−	×	−	×	4	11	7	0
48	−	−	×	−	5	15	9	−1
50	−	−	−	−	6	15	9	0
56	−	×	−	−	7	16	9	0
60	−	×	×	×	6	17	10	−1
$C1$	×	×	×	×	−	−	−	−
$C2$	×	×	×	−	−	−	−	−
$C3$	×	×	−	×	−	−	−	−
$C4$	×	×	−	−	−	−	−	−
$C5$	×	−	×	×	−	−	−	−

ii) Let us demonstrate by recurrence that if (11) is true for n, it is true for $n+2$ too.

Let us make the hypothesis that property 11 is true for n,

$$Y_a(n) = \pi(n) - \pi\left(\frac{n}{2}\right) + \delta_{Y_a}(n) \qquad (H)$$

Let us demonstrate it is true for $n+2$,

$$Y_a(n+2) = \pi(n+2) - \pi\left(\frac{n+2}{2}\right) + \delta_{Y_a}(n+2) \qquad (Ccl)$$

Let us study the 11 cases listed in last section.

iia) $n+1$ compound, $(n+2)/2$ compound, $n/2$ prime, $n-1$ prime

In that case, $\pi(n+2) = \pi(n)$, $\pi\left(\frac{n+2}{2}\right) = \pi\left(\frac{n}{2}\right)$, $\delta_{Y_a}(n) = 0$ and $Y_a(n+2) = Y_a(n)$.

So we have,

$$\begin{aligned} Y_a(n+2) &= Y_a(n) \\ &= \pi(n) - \pi\left(\frac{n}{2}\right) + \delta_{Y_a}(n) \\ &= \pi(n+2) - \pi\left(\frac{n+2}{2}\right) + \delta_{Y_a}(n+2) \end{aligned}$$

since $\delta_{Y_a}(n+2) = 0$ in that case. It implies that (Ccl) is verified.

iib) $n+1$ prime, $(n+2)/2$ compound, $n/2$ compound, $n-1$ compound

In that case, $\pi(n+2) = \pi(n) + 1$, $\pi\left(\frac{n+2}{2}\right) = \pi\left(\frac{n}{2}\right)$, $\delta_{Y_a}(n) = 0$ and $Y_a(n+2) = Y_a(n)$.

So we have,

$$\begin{aligned} Y_a(n+2) &= Y_a(n) \\ &= \pi(n) - \pi\left(\frac{n}{2}\right) + \delta_{Y_a}(n) \\ &= \pi(n+2) - 1 - \pi\left(\frac{n+2}{2}\right) + \delta_{Y_a}(n+2) \end{aligned}$$

since $\delta_{Y_a}(n+2) = -1$ in that case. This implies that (Ccl) is verified.

iic) $n+1$ compound, $(n+2)/2$ compound, $n/2$ prime, $n-1$ compound

In that case, $\pi(n+2) = \pi(n)$, $\pi\left(\frac{n+2}{2}\right) = \pi\left(\frac{n}{2}\right)$, $\delta_{Y_a}(n) = 1$ and $Y_a(n+2) = Y_a(n) - 1$.

So we have,

$$\begin{aligned} Y_a(n+2) &= Y_a(n) - 1 \\ &= \pi(n) - \pi\left(\frac{n}{2}\right) + \delta_{Y_a}(n) - 1 \\ &= \pi(n+2) - \pi\left(\frac{n+2}{2}\right) + 1 - 1 \\ &= \pi(n+2) - \pi\left(\frac{n+2}{2}\right) + \delta_{Y_a}(n+2) \end{aligned}$$

since $\delta_{Y_a}(n+2) = 0$ in that case. This implies that (Ccl) is verified.

iid) $n+1$ prime, $(n+2)/2$ prime, $n/2$ compound, $n-1$ prime

In that case, $\pi(n+2) = \pi(n) + 1$, $\pi\left(\frac{n+2}{2}\right) = \pi\left(\frac{n}{2}\right) + 1$, $\delta_{Y_a}(n) = -1$ and $Y_a(n+2) = Y_a(n) + 1$.

So we have,

$$\begin{aligned} Y_a(n+2) &= Y_a(n) + 1 \\ &= \pi(n) - \pi\left(\frac{n}{2}\right) + \delta_{Y_a}(n) + 1 \\ &= \pi(n+2) - 1 - \pi\left(\frac{n+2}{2}\right) + 1 - 1 + 1 \\ &= \pi(n+2) - \pi\left(\frac{n+2}{2}\right) + \delta_{Y_a}(n+2) \end{aligned}$$

since $\delta_{Y_a}(n+2) = 0$ in that case. This implies that (Ccl) is verified.

iie) $n+1$ compound, $(n+2)/2$ prime, $n/2$ compound, $n-1$ prime

In that case, $\pi(n+2) = \pi(n)$, $\pi\left(\frac{n+2}{2}\right) = \pi\left(\frac{n}{2}\right) + 1$, $\delta_{Y_a}(n) = -1$ and $Y_a(n+2) = Y_a(n) + 1$.

So we have,

$$\begin{aligned} Y_a(n+2) &= Y_a(n) + 1 \\ &= \pi(n) - \pi\left(\frac{n}{2}\right) + \delta_{Y_a}(n) + 1 \\ &= \pi(n+2) - \pi\left(\frac{n+2}{2}\right) + 1 - 1 + 1 \\ &= \pi(n+2) - \pi\left(\frac{n+2}{2}\right) + \delta_{Y_a}(n+2) \end{aligned}$$

since $\delta_{Y_a}(n+2) = 1$ in that case. This implies that (Ccl) is verified.

iif) $n+1$ prime, $(n+2)/2$ prime, $n/2$ compound, $n-1$ compound

In that case, $\pi(n+2) = \pi(n) + 1$, $\pi\left(\frac{n+2}{2}\right) = \pi\left(\frac{n}{2}\right) + 1$, $\delta_{Y_a}(n) = 0$ and $Y_a(n+2) = Y_a(n)$.

So we have,

$$\begin{aligned} Y_a(n+2) &= Y_a(n) \\ &= \pi(n) - \pi\left(\frac{n}{2}\right) + \delta_{Y_a}(n) \\ &= \pi(n+2) - 1 - \pi\left(\frac{n+2}{2}\right) + 1 \\ &= \pi(n+2) - \pi\left(\frac{n+2}{2}\right) + \delta_{Y_a}(n+2) \end{aligned}$$

since $\delta_{Y_a}(n+2) = 0$ in that case. This implies that (Ccl) is verified.

iig) $n+1$ compound, $(n+2)/2$ prime, $n/2$ compound, $n-1$ compound

In that case, $\pi(n+2) = \pi(n)$, $\pi\left(\frac{n+2}{2}\right) = \pi\left(\frac{n}{2}\right) + 1$, $\delta_{Y_a}(n) = 0$ and $Y_a(n+2) = Y_a(n)$.

So we have,

$$\begin{aligned}
Y_a(n+2) &= Y_a(n) \\
&= \pi(n) - \pi\left(\frac{n}{2}\right) + \delta_{Y_a}(n) \\
&= \pi(n+2) - \pi\left(\frac{n+2}{2}\right) + 1 + 0 \\
&= \pi(n+2) - \pi\left(\frac{n+2}{2}\right) + \delta_{Y_a}(n+2)
\end{aligned}$$

since $\delta_{Y_a}(n+2) = 1$ in that case. This implies that (Ccl) is verified.

iih) $n+1$ prime, $(n+2)/2$ compound, $n/2$ compound, $n-1$ prime

In that case, $\pi(n+2) = \pi(n) + 1$, $\pi\left(\frac{n+2}{2}\right) = \pi\left(\frac{n}{2}\right)$, $\delta_{Y_a}(n) = -1$ and $Y_a(n+2) = Y_a(n) + 1$.

So we have,

$$\begin{aligned}
Y_a(n+2) &= Y_a(n) + 1 \\
&= \pi(n) - \pi\left(\frac{n}{2}\right) + \delta_{Y_a}(n) + 1 \\
&= \pi(n+2) - 1 - \pi\left(\frac{n+2}{2}\right) - 1 + 1 \\
&= \pi(n+2) - \pi\left(\frac{n+2}{2}\right) + \delta_{Y_a}(n+2)
\end{aligned}$$

since $\delta_{Y_a}(n+2) = -1$ in that case. This implies that (Ccl) is verified.

iii) $n+1$ compound, $(n+2)/2$ compound, $n/2$ compound, $n-1$ prime

In that case, $\pi(n+2) = \pi(n)$, $\pi\left(\frac{n+2}{2}\right) = \pi\left(\frac{n}{2}\right)$, $\delta_{Y_a}(n) = -1$ and $Y_a(n+2) = Y_a(n) + 1$.

So we have,

$$\begin{aligned}
Y_a(n+2) &= Y_a(n) + 1 \\
&= \pi(n) - \pi\left(\frac{n}{2}\right) + \delta_{Y_a}(n) + 1 \\
&= \pi(n+2) - \pi\left(\frac{n+2}{2}\right) - 1 + 1 \\
&= \pi(n+2) - \pi\left(\frac{n+2}{2}\right) + \delta_{Y_a}(n+2)
\end{aligned}$$

since $\delta_{Y_a}(n+2) = 0$ in that case. It implies that (Ccl) is verified.

iij) $n+1$ prime, $(n+2)/2$ compound, $n/2$ prime, $n-1$ compound

In that case, $\pi(n+2) = \pi(n) + 1$, $\pi\left(\frac{n+2}{2}\right) = \pi\left(\frac{n}{2}\right)$, $\delta_{Y_a}(n) = 1$ and $Y_a(n+2) = Y_a(n) - 1$.

So we have,

$$\begin{aligned}
Y_a(n+2) &= Y_a(n) - 1 \\
&= \pi(n) - \pi\left(\frac{n}{2}\right) + \delta_{Y_a}(n) - 1 \\
&= \pi(n+2) - 1 - \pi\left(\frac{n+2}{2}\right) + 1 - 1 \\
&= \pi(n+2) - \pi\left(\frac{n+2}{2}\right) + \delta_{Y_a}(n+2)
\end{aligned}$$

since $\delta_{Y_a}(n+2) = -1$ in that case. This implies that (Ccl) is verified.

iik) $n+1$ compound, $(n+2)/2$ compound, $n/2$ compound, $n-1$ compound

In that case, $\pi(n+2) = \pi(n)$, $\pi\left(\frac{n+2}{2}\right) = \pi\left(\frac{n}{2}\right)$, $\delta_{Y_a}(n) = 0$ and $Y_a(n+2) = Y_a(n)$.

So we have,

$$\begin{aligned}
Y_a(n+2) &= Y_a(n) \\
&= \pi(n) - \pi\left(\frac{n}{2}\right) + \delta_{Y_a}(n) \\
&= \pi(n+2) - \pi\left(\frac{n+2}{2}\right) + \delta_{Y_a}(n+2)
\end{aligned}$$

since $\delta_{Y_a}(n+2) = 0$ in that case. This implies that (Ccl) is verified.

6.1.4 Property 12 : invariant relation concerning $Y_c(n)$

The objective is to demonstrate why the invariant relation concerning $Y_c(n)$ and that is :

$$Y_c(n) = \left\lfloor \frac{n}{4} \right\rfloor - \pi(n) + \pi\left(\frac{n}{2}\right) + \delta_{Y_c}(n)$$

is always verified.

i) recurrences'initializations :

n	$n/2$ *is prime*	$(n+2)/2$ *is prime*	$n-1$ *is prime*	$n+1$ *is prime*	$Y_c(n)$	$\left\lfloor \frac{n}{4} \right\rfloor$	$\pi(n)$	$\pi(n/2)$	$\delta_{Y_c}(n)$
14	×	−	×	−	1	3	6	4	0
16	−	−	−	×	1	4	6	4	−1
18	−	−	×	×	2	4	7	4	1
20	−	×	×	−	1	5	8	4	0
22	×	−	−	×	1	5	8	5	−1
26	×	−	−	−	2	6	9	6	−1
36	−	×	−	×	4	9	11	7	−1
48	−	−	×	−	6	12	15	9	0
50	−	−	−	−	6	12	15	9	0
56	−	×	−	−	6	14	16	9	−1
60	−	×	×	×	8	15	17	10	0

It is easy to verify that for small integers, property 12 is well systematically verified.

ii) Let us demonstrate by recurrence that if (12) is true for n, it is true for $n+2$.

Let us make the hypothesis that property 12 is true for n,

$$Y_c(n) = \left\lfloor \frac{n}{4} \right\rfloor - \pi(n) + \pi\left(\frac{n}{2}\right) + \delta_{Y_c}(n) \qquad (H)$$

Let us demonstrate it is true for $n+2$,

$$Y_c(n+2) = \left\lfloor \frac{n+2}{4} \right\rfloor - \pi(n+2) + \pi\left(\frac{n+2}{2}\right) + \delta_{Y_c}(n+2) \qquad (Ccl)$$

We find here another time the 11 cases identified for demonstrating property 11 ; sometimes, there are two subcases, when $n/2$ is compound, according to the fact that n is an even's double in which case $\left\lfloor \frac{n+2}{4} \right\rfloor = \left\lfloor \frac{n}{4} \right\rfloor$ or that n is an odd's double $\left\lfloor \frac{n+2}{4} \right\rfloor = \left\lfloor \frac{n}{4} \right\rfloor + 1$.

In the array below are provided $\delta_{Y_c}(n)$'s values according to the booleans that must be considered (in first column, we indicate a number of line l_i that will be useful in the following ; an example of a small integer is provided in penultimate column, to fix ideas ; the letter corresponding to the recurrence type intervening (cf after the array) is provided in last column :

	$n-1$ *is prime*	$n+1$ *is prime*	$n/2$ *is prime*	$(n+2)/2$ *is prime*	$\delta_{4k+2}(n)$	$\delta_{Y_c}(n)$	*ex*	*case*
l_1	×	–	×	–	×	0	14	a
l_2	×	–	–	×	–	0	20	e
l_3	×	–	–	–	×	0	54	i
l_4	×	–	–	–	–	1	48	i
l_5	–	×	×	–	×	-1	22	b
l_6	–	×	–	×	–	-1	36	f
l_7	–	×	–	–	×	0	66	j
l_8	–	×	–	–	–	-1	16	j
l_9	–	–	×	–	×	-1	26	c
l_{10}	–	–	–	×	–	-1	56	g
l_{11}	–	–	–	–	×	0	50	k
l_{12}	–	–	–	–	–	-1	64	k
l_{13}	×	×	–	×	–	0	60	d
l_{14}	×	×	–	–	×	1	18	h
l_{15}	×	×	–	–	–	0	108	h

iia) $n+1$ compound, $(n+2)/2$ compound, $n/2$ prime, $n-1$ prime

In that case, $\pi(n+2)=\pi(n)$, $\pi\left(\dfrac{n+2}{2}\right)=\pi\left(\dfrac{n}{2}\right)$, $\delta_{Y_c}(n)=0$ and $Y_c(n+2)=Y_c(n)$.

So we have,

$$\begin{aligned}
Y_c(n+2) &= Y_c(n)\\
&= \left\lfloor\frac{n}{4}\right\rfloor-\pi(n)+\pi\left(\frac{n}{2}\right)+\delta_{Y_c}(n)\\
&= \left\lfloor\frac{n+2}{4}\right\rfloor-1-\pi(n+2)+\pi\left(\frac{n+2}{2}\right)+0\\
&= \left\lfloor\frac{n+2}{4}\right\rfloor-\pi(n+2)+\pi\left(\frac{n+2}{2}\right)+\delta_{Y_c}(n+2)
\end{aligned}$$

since $\delta_{Y_c}(n+2)=-1$ in that case ($n+2$ corresponds than to one of the cases of lines l_6, l_8, l_{10} or l_{12} because n being an odd's double, $n+2$ can't be one). This implies that (Ccl) is verified.

iib) $n+1$ prime, $(n+2)/2$ compound, $n/2$ prime, $n-1$ compound

In that case, $\pi(n+2)=\pi(n)+1$, $\pi\left(\dfrac{n+2}{2}\right)=\pi\left(\dfrac{n}{2}\right)$, $\delta_{Y_c}(n)=-1$ and $Y_c(n+2)=Y_c(n)+1$.

So we have,

$$\begin{aligned}
Y_c(n+2) &= Y_c(n)+1\\
&= \left\lfloor\frac{n}{4}\right\rfloor-\pi(n)+\pi\left(\frac{n}{2}\right)+\delta_{Y_c}(n)+1\\
&= \left\lfloor\frac{n+2}{4}\right\rfloor-1-\pi(n+2)+1+\pi\left(\frac{n+2}{2}\right)-1+1\\
&= \left\lfloor\frac{n+2}{4}\right\rfloor-\pi(n+2)+\pi\left(\frac{n+2}{2}\right)+\delta_{Y_c}(n+2)
\end{aligned}$$

since $\delta_{Y_c}(n+2)=0$ in that case ($n+2$ corresponds then to one of the cases of lines l_2, l_{13} or l_{15} because n being an odd's double, $n+2$ can't be one). This implies that (Ccl) is verified.

iic) $n+1$ compound, $(n+2)/2$ compound, $n/2$ prime, $n-1$ compound

In that case, $\pi(n+2)=\pi(n)$, $\pi\left(\dfrac{n+2}{2}\right)=\pi\left(\dfrac{n}{2}\right)$, $\delta_{Y_c}(n)=-1$ and $Y_c(n+2)=Y_c(n)+1$.

So we have,

$$\begin{aligned}
Y_c(n+2) &= Y_c(n)+1\\
&= \left\lfloor\frac{n}{4}\right\rfloor-\pi(n)+\pi\left(\frac{n}{2}\right)+\delta_{Y_c}(n)+1\\
&= \left\lfloor\frac{n+2}{4}\right\rfloor-1-\pi(n+2)+\pi\left(\frac{n+2}{2}\right)-1+1\\
&= \left\lfloor\frac{n+2}{4}\right\rfloor-\pi(n+2)+\pi\left(\frac{n+2}{2}\right)+\delta_{Y_c}(n+2)
\end{aligned}$$

since $\delta_{Y_c}(n+2) = -1$ in that case ($n+2$ corresponds then to one of the cases of lines l_6, l_8, l_{10} or l_{12} because n being an odd's double, $n+2$ can't be one). This implies that (Ccl) is verified.

iid) $n+1$ prime, $(n+2)/2$ prime, $n/2$ compound, $n-1$ prime

In that case, $\pi(n+2) = \pi(n)+1$, $\pi\left(\dfrac{n+2}{2}\right) = \pi\left(\dfrac{n}{2}\right)+1$, $\delta_{Y_c}(n) = 0$ and $Y_c(n+2) = Y_c(n)$.

n is mandatory an even's double.

So we have,

$$\begin{aligned} Y_c(n+2) &= Y_c(n) \\ &= \left\lfloor \frac{n}{4} \right\rfloor - \pi(n) + \pi\left(\frac{n}{2}\right) + \delta_{Y_c}(n) \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi(n+2) + 1 + \pi\left(\frac{n+2}{2}\right) - 1 + 0 \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi(n+2) + \pi\left(\frac{n+2}{2}\right) + \delta_{Y_c}(n+2) \end{aligned}$$

since $\delta_{Y_c}(n+2) = 0$ in that case ($n+2$ corresponds then to the case of line l_1). This implies that (Ccl) is verified.

iie) $n+1$ compound, $(n+2)/2$ prime, $n/2$ compound, $n-1$ prime

In that case, $\pi(n+2) = \pi(n)$, $\pi\left(\dfrac{n+2}{2}\right) = \pi\left(\dfrac{n}{2}\right)+1$, $\delta_{Y_c}(n) = 0$ and $Y_c(n+2) = Y_c(n)$.

n is mandatory an even's double.

So we have,

$$\begin{aligned} Y_c(n+2) &= Y_c(n) \\ &= \left\lfloor \frac{n}{4} \right\rfloor - \pi(n) + \pi\left(\frac{n}{2}\right) + \delta_{Y_c}(n) \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi(n+2) + \pi\left(\frac{n+2}{2}\right) - 1 + 0 \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi(n+2) + \pi\left(\frac{n+2}{2}\right) + \delta_{Y_c}(n+2) \end{aligned}$$

since $\delta_{Y_c}(n+2) = -1$ in that case ($n+2$ corresponds then to one of the cases of lines l_5 or l_9). This implies that (Ccl) is verified.

iif) $n+1$ prime, $(n+2)/2$ prime, $n/2$ compound, $n-1$ compound

In that case, $\pi(n+2) = \pi(n)+1$, $\pi\left(\dfrac{n+2}{2}\right) = \pi\left(\dfrac{n}{2}\right)+1$, $\delta_{Y_c}(n) = -1$ and $Y_c(n+2) = Y_c(n)+1$.

n is mandatory an even's double.

So we have,

$$\begin{aligned} Y_c(n+2) &= Y_c(n)+1 \\ &= \left\lfloor \frac{n}{4} \right\rfloor - \pi(n) + \pi\left(\frac{n}{2}\right) + \delta_{Y_c}(n) + 1 \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi(n+2) + 1 + \pi\left(\frac{n+2}{2}\right) - 1 - 1 + 1 \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi(n+2) + \pi\left(\frac{n+2}{2}\right) + \delta_{Y_c}(n+2) \end{aligned}$$

since $\delta_{Y_c}(n+2) = 0$ in that case ($n+2$ corresponds then to the case of line l_1). This implies that (Ccl) is verified.

iig) $n+1$ compound, $(n+2)/2$ prime, $n/2$ compound, $n-1$ compound

In that case, $\pi(n+2) = \pi(n)$, $\pi\left(\dfrac{n+2}{2}\right) = \pi\left(\dfrac{n}{2}\right)+1$, $\delta_{Y_c}(n) = -1$ and $Y_c(n+2) = Y_c(n)+1$.

n is mandatory an even's double.

So we have,

$$\begin{aligned} Y_c(n+2) &= Y_c(n)+1 \\ &= \left\lfloor \frac{n}{4} \right\rfloor - \pi(n) + \pi\left(\frac{n}{2}\right) + \delta_{Y_c}(n) + 1 \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi(n+2) + \pi\left(\frac{n+2}{2}\right) - 1 - 1 + 1 \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi(n+2) + \pi\left(\frac{n+2}{2}\right) + \delta_{Y_c}(n+2) \end{aligned}$$

since $\delta_{Y_c}(n+2) = -1$ in that case ($n+2$ corresponds then to one of the cases of lines l_5 or l_9). This implies that (Ccl) is verified.

iih) $n+1$ prime, $(n+2)/2$ compound, $n/2$ compound, $n-1$ prime

In that case, $\pi(n+2) = \pi(n)+1$, $\pi\left(\frac{n+2}{2}\right) = \pi\left(\frac{n}{2}\right)$.

iih1) n even's double, $\delta_{Y_c}(n) = 0$ and $Y_c(n+2) = Y_c(n)$.

So we have,

$$\begin{aligned} Y_c(n+2) &= Y_c(n) \\ &= \left\lfloor \frac{n}{4} \right\rfloor - \pi(n) + \pi\left(\frac{n}{2}\right) + \delta_{Y_c}(n) \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi(n+2) + 1 + \pi\left(\frac{n+2}{2}\right) + 0 \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi(n+2) + \pi\left(\frac{n+2}{2}\right) + \delta_{Y_c}(n+2) \end{aligned}$$

since $\delta_{Y_c}(n+2) = 1$ in that case ($n+2$ corresponds then to one of the cases of lines l_3 or l_{14}). This implies that (Ccl) is verified.

iih2) n odd's double, $\delta_{Y_c}(n) = 1$ and $Y_c(n+2) = Y_c(n) - 1$.

So we have,

$$\begin{aligned} Y_c(n+2) &= Y_c(n) - 1 \\ &= \left\lfloor \frac{n}{4} \right\rfloor - \pi(n) + \pi\left(\frac{n}{2}\right) + \delta_{Y_c}(n) - 1 \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - 1 - \pi(n+2) + 1 + \pi\left(\frac{n+2}{2}\right) + 1 - 1 \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi(n+2) + \pi\left(\frac{n+2}{2}\right) + \delta_{Y_c}(n+2) \end{aligned}$$

since $\delta_{Y_c}(n+2) = 0$ in that case ($n+2$ corresponds then to one of the cases of lines l_2, l_4, l_{13} or l_{15}). This implies that (Ccl) is verified.

iii) $n+1$ compound, $(n+2)/2$ compound, $n/2$ compound, $n-1$ prime

In that case, $\pi(n+2) = \pi(n)$, $\pi\left(\frac{n+2}{2}\right) = \pi\left(\frac{n}{2}\right)$.

iii1) n even's double, $\delta_{Y_c}(n) = 0$ and $Y_c(n+2) = Y_c(n)$.

So we have,

$$\begin{aligned} Y_c(n+2) &= Y_c(n) \\ &= \left\lfloor \frac{n}{4} \right\rfloor - \pi(n) + \pi\left(\frac{n}{2}\right) + \delta_{Y_c}(n) \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi(n+2) + \pi\left(\frac{n+2}{2}\right) + 0 \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi(n+2) + \pi\left(\frac{n+2}{2}\right) + \delta_{Y_c}(n+2) \end{aligned}$$

since $\delta_{Y_c}(n+2) = 0$ in that case ($n+2$ corresponds then to one of the cases of lines l_7 or l_{11}). This implies that (Ccl) is verified.

iii2) n odd's double, $\delta_{Y_c}(n) = 1$ and $Y_c(n+2) = Y_c(n) - 1$.

So we have,

$$\begin{aligned}
Y_c(n+2) &= Y_c(n) - 1 \\
&= \left\lfloor \frac{n}{4} \right\rfloor - \pi(n) + \pi\left(\frac{n}{2}\right) + \delta_{Y_c}(n) - 1 \\
&= \left\lfloor \frac{n+2}{4} \right\rfloor - 1 - \pi(n+2) + \pi\left(\frac{n+2}{2}\right) + 1 - 1 \\
&= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi(n+2) + \pi\left(\frac{n+2}{2}\right) + \delta_{Y_c}(n+2)
\end{aligned}$$

since $\delta_{Y_c}(n+2) = -1$ in that case ($n+2$ corresponds then to one of the cases of lines l_6, l_8, l_{10} or l_{12}). This implies that (Ccl) is verified.

iij) $n+1$ prime, $(n+2)/2$ compound, $n/2$ compound, $n-1$ compound

In that case, $\pi(n+2) = \pi(n) + 1$, $\pi\left(\frac{n+2}{2}\right) = \pi\left(\frac{n}{2}\right)$.

iij1) n even's double, $\delta_{Y_c}(n) = -1$ and $Y_c(n+2) = Y_c(n) + 1$.

So we have,

$$\begin{aligned}
Y_c(n+2) &= Y_c(n) + 1 \\
&= \left\lfloor \frac{n}{4} \right\rfloor - \pi(n) + \pi\left(\frac{n}{2}\right) + \delta_{Y_c}(n) + 1 \\
&= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi(n+2) + 1 + \pi\left(\frac{n+2}{2}\right) - 1 + 1 \\
&= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi(n+2) + \pi\left(\frac{n+2}{2}\right) + \delta_{Y_c}(n+2)
\end{aligned}$$

since $\delta_{Y_c}(n+2) = 1$ in that case ($n+2$ corresponds then to one of the cases of lines l_3 or l_{14}). This implies that (Ccl) is verified.

iij2) n odd's double, $\delta_{Y_c}(n) = 0$ and $Y_c(n+2) = Y_c(n)$.

So we have,

$$\begin{aligned}
Y_c(n+2) &= Y_c(n) \\
&= \left\lfloor \frac{n}{4} \right\rfloor - \pi(n) + \pi\left(\frac{n}{2}\right) + \delta_{Y_c}(n) \\
&= \left\lfloor \frac{n+2}{4} \right\rfloor - 1 - \pi(n+2) + 1 + \pi\left(\frac{n+2}{2}\right) + 0 \\
&= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi(n+2) + \pi\left(\frac{n+2}{2}\right) + \delta_{Y_c}(n+2)
\end{aligned}$$

since $\delta_{Y_c}(n+2) = 0$ in that case ($n+2$ corresponds then to one of the cases of lines l_2, l_4, l_{13} or l_{15}). This implies that (Ccl) is verified.

iik) $n+1$ compound, $(n+2)/2$ compound, $n/2$ compound, $n-1$ compound

In that case, $\pi(n+2) = \pi(n)$, $\pi\left(\frac{n+2}{2}\right) = \pi\left(\frac{n}{2}\right)$.

iik1) n even's double, $\delta_{Y_c}(n) = -1$ and $Y_c(n+2) = Y_c(n) + 1$.

So we have,

$$\begin{aligned}
Y_c(n+2) &= Y_c(n) + 1 \\
&= \left\lfloor \frac{n}{4} \right\rfloor - \pi(n) + \pi\left(\frac{n}{2}\right) + \delta_{Y_c}(n) + 1 \\
&= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi(n+2) + \pi\left(\frac{n+2}{2}\right) - 1 + 1 \\
&= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi(n+2) + \pi\left(\frac{n+2}{2}\right) + \delta_{Y_c}(n+2)
\end{aligned}$$

since $\delta_{Y_c}(n+2) = 0$ in that case ($n+2$ corresponds then to one of the cases of lines l_7 or l_{11}). This implies that (Ccl) is verified.

iik2) n odd compound's double, $\delta_{Y_c}(n) = 0$ and $Y_c(n+2) = Y_c(n)$

So we have,

$$\begin{aligned} Y_c(n+2) &= Y_c(n) \\ &= \left\lfloor \frac{n}{4} \right\rfloor - \pi(n) + \pi\left(\frac{n}{2}\right) + \delta_{Y_c}(n) \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - 1 - \pi(n+2) + \pi\left(\frac{n+2}{2}\right) + 0 \\ &= \left\lfloor \frac{n+2}{4} \right\rfloor - \pi(n+2) + \pi\left(\frac{n+2}{2}\right) + \delta_{Y_c}(n+2) \end{aligned}$$

since $\delta_{Y_c}(n+2) = -1$ in that case ($n+2$ corresponds to cases in lines l_6, l_8, l_{10} or l_{12}). This implies that (Ccl) is verified.

6.2 Order relations between variables

6.2.1 Study of inequalities $Z_c(n) > Z_a(n), Z_a(n) > Y_a(n)$ and $Y_c(n) > Z_c(n)$

From properties 9 and 10, we deduce that $Z_c(n) - Z_a(n) = \left\lfloor \frac{n}{4} \right\rfloor - 2\,\pi\left(\frac{n}{2}\right) + \delta_{Z_c - Z_a}(n)$ with $\delta_{Z_c - Z_a}(n)$ equal to $1, 2$ or 3.

$Z_c(n) - Z_a(n)$ seems globally increasing with little variations. One observes that $Z_c(n) > Z_a(n)$ for $n \geqslant 240$. For greater values, strict inequality will be always verified because $\left\lfloor \frac{n}{4} \right\rfloor$ is increased much more often than $2\,\pi\left(\frac{n}{2}\right)$.

From properties 9 and 11, we deduce that $Z_a(n) - Y_a(n) = 2\,\pi\left(\frac{n}{2}\right) - \pi(n) + \delta_{Z_a - Y_a}(n)$ with $\delta_{Z_a - Y_a}(n)$ equal to $-3, -2, -1$ or 0.

$Z_a(n) - Y_a(n)$ seems globally increasing with little variations. One observes that $Z_a(n) > Y_a(n)$ for $n \geqslant 36$.

From properties 10 and 12, we deduce that $Y_c(n) - Z_c(n) = 2\,\pi\left(\frac{n}{2}\right) - \pi(n) + \delta_{Y_c - Z_c}(n)$ with $\delta_{Y_c - Z_c}(n)$ equal to $-2, -1, 0$ or 1.

$Y_c(n) - Z_c(n)$ is an increasing function of n. One observes that $Y_c(n) > Z_c(n)$ for $n \geqslant 24$.

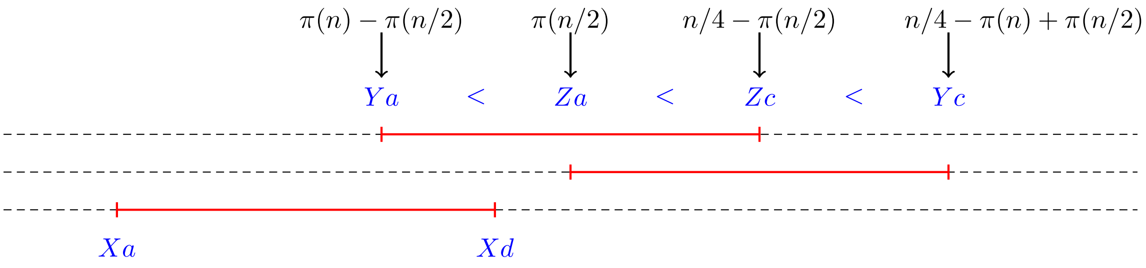
6.2.2 Strict order between 4 variables $Y_a(n), Y_c(n), Z_a(n)$ and $Z_c(n)$

Variables $Y_a(n), Z_a(n), Z_c(n)$ et $Y_c(n)$ are strictly ordered in the following way :

$$Y_a(n) < Z_a(n) < Z_c(n) < Y_c(n)$$

for all $n \geqslant 240$.

A picture showing variables'gaps is provided below, that illustrates their intrication :



From properties 9, 10, 11 and 12, we deduce :

- $Z_c(n) - Y_a(n) = \left\lfloor \frac{n}{4} \right\rfloor - \pi(n) + \delta_{Z_c - Y_a}(n)$ with $\delta_{Z_c - Y_a}(n)$ equal to $-1, 0, 1$ or 2 ;
- $Y_c(n) - Z_a(n) = \left\lfloor \frac{n}{4} \right\rfloor - \pi(n) + \delta_{Y_c - Z_a}(n)$ with $\delta_{Y_c - Z_a}(n)$ equal to $0, 1, 2$ or 3 ;
- $X_d(n) - X_a(n) = Z_c(n) - Y_a(n) + \delta_{2odd-compound}(n) = \left\lfloor \frac{n}{4} \right\rfloor - \pi(n) + \delta_{X_d - X_a}(n)$ with $\delta_{X_d - X_a}(n)$ equal to $-1, 0, 1$ or 2.

One can also observe by program that :

- $X_c(n) - X_b(n) = 2\pi\left(\frac{n}{2}\right) - \pi(n) + \delta_{X_c - X_b}(n)$ with $\delta_{X_c - X_b}(n)$ equal to $-2, -1, 0$ or 1.

Also, one can deduce from equalities yet provided that :

- $X_a(n) + X_b(n) = \pi(n) - \pi\left(\frac{n}{2}\right) + \delta_{X_c - X_b}(n)$ with $\delta_{X_a + X_b}(n)$ very small.

The function $\left\lfloor \frac{n}{4} \right\rfloor - \pi(n)$ that appear in right members of three first equalities above is globally increasing with little variations. It's equal to 0 for $n = 122$.

Since $\left\lfloor \frac{n+2}{4} \right\rfloor = \left\lfloor \frac{n}{4} \right\rfloor + 1$ when n is an even's double (i.e. for one even number each two) while $\pi(n+2) = \pi(n) + 1$ when $n+1$ is prime (far less often than for one even number each two), for all n greater than a small value (i.e. $n > 122$), we will always have $\left\lfloor \frac{n}{4} \right\rfloor - \pi(n) > 0$.

6.3 Study of inequality $X_a(n) > 0$

To be sure that $X_a(n)$ would never be equal to 0, we should have to show that since a certain value of n, the inequality

$$X_d(n) > \left\lfloor \frac{n}{4} \right\rfloor - \pi(n) + 2$$

is always verified.

Indeed, this inequality, combined with the invariant $X_d(n) - X_a(n) = \left\lfloor \frac{n}{4} \right\rfloor - \pi(n) + \delta_{X_d - X_a}(n)$, would have as consequence that $X_a(n)$ would be always strictly positive.

We observe that it seems that

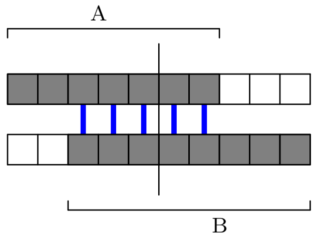
$$X_d(n) > \left\lfloor \frac{n}{4} \right\rfloor - \pi(n) + 2$$

for $n \geqslant 30$.

Intuitively, one understands what process is at work : the number of compound numbers becomes so big comparatively to the number of prime numbers that each compound number has "many more chances" to be the complementary to n of another compound number rather than a prime one. It is thus one can observe $X_d(n)$ becomes quickly greater than the sum $X_b(n) + X_c(n)$, in which each term is greater than $X_a(n)$.

6.4 Subsidiary exercize

Let us propose as an exercize that we must match (with a bijection) numbers of two sets ; in the first set containing k numbers, A are compound and $k - A$ are prime while in the second set, containing also k numbers, B numbers are compound while $k - B$ are prime. If $A \leqslant k/2$ and $B \leqslant k/2$, the matching can put in bijection two numbers that are not mandatory compound. But as soon as $A > k/2$ and $B > k/2$, some matching must necessarily put in bijection a compound number from the first set with a compound number of the second set. We can easily understand thanks to the drawing below that the number of matchings associating bijectively two compound numbers is necessarily greater than $A + B - k$.



6.5 $X_a(n)$ constrained to be strictly positive

Among odd numbers that are between 3 and $n/2$, very quickly, more than a half are compound. On the same manner, among odd numbers that are between $n/2$ and $n-3$, very quickly, more than a half are compound.

By this way, there are rather quickly both more than a half of numbers that can be an n's decomposition first range sommant that are compound and more than a half of numbers that can be an n's decomposition second range sommant that are compound too (i.e. $Y_c(n) > \left\lfloor \frac{n-2}{8} \right\rfloor$ and $Z_c(n) > \left\lfloor \frac{n-2}{8} \right\rfloor$) as soon as $n \geqslant 244$. Indeed, for $n \geqslant 244$, $\left\lfloor \frac{n}{4} \right\rfloor - \pi(n) + \pi\left(\frac{n}{2}\right) > \left\lfloor \frac{n-2}{8} \right\rfloor$ and $\left\lfloor \frac{n}{4} \right\rfloor - \pi\left(\frac{n}{2}\right) > \left\lfloor \frac{n-2}{8} \right\rfloor$. $X_d(n)$ counting n's decompositions of the form *compound*+*compound* is then systematically greater than $Y_c(n) + Z_c(n) - \left\lfloor \frac{n-4}{4} \right\rfloor$ (cf subsidiary exercize of precedent section) and will remain greater than it definitively, which ensures that $X_a(n)$ will be always strictly positive, which proves binary Goldbach's conjecture.