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Asymptotically Abelian II_1-Factors

By

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1. Introduction

Recently, physicists (cf. [2], [4], [5], [7], [8], [12], [13], [14]) have introduced the notion of asymptotic abelianness into the theory of operator algebras and obtained various interesting results. In the present paper, we shall extend this notion to finite factors, and by using it, we shall show the existence of a new II_1-factor. For type III-factors, we shall discuss in another paper.

2. Theorems

First of all, we shall define

Definition 1. Let M be a finite factor and let τ be the unique normalized trace on M. M is called asymptotically abelian, if there exists a sequence of *-automorphisms $\{\rho_n\}$ on M such that $\lim_{n\to\infty} ||[\rho_n(a), b]||_2=0$ for $a, b\in M$, where $[x, y]=xy-yx$ and $||x||_2=\tau(x^*x)^{1/2}$ for $x, y\in M$.

Let $\mathfrak{A}$ be a finite factor, and let φ be the normalized trace on $\mathfrak{A}$. Let $\mathfrak{L}=\bigotimes_{n=1}^{\infty}\mathfrak{A}_n$ with $\mathfrak{A}_n=\mathfrak{A}$ be the infinite C^*-tensor product (cf. [3]), and let $\psi=\bigotimes_{n=1}^{\infty}\varphi_n$ with $\varphi_n=\varphi$ be the infinite product trace on $\mathfrak{L}$. Let G be the group of finite permutations of positive integers N, i.e. an element $g\in G$ is a one-to-one mapping of N onto itself which leaves all but a finite number of positive integers fixed.

Then, g will define a *-automorphism, also denoted by g of $\mathfrak{L}$

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by $g(\Sigma\otimes a_n)=\Sigma\otimes a_{g(n)}$, where $a_n=1$ for all but a finite number of indices.

For each integer n, we denote by g_n the permutation

$$g_n(k)=\begin{cases} 2^{n-1}+k & \text{if } 1\leqq k\leqq 2^{n-1} \\ k-2^{n-1} & \text{if } 2^{n-1}<k\leqq 2^n \\ k & \text{if } 2^n<k\,. \end{cases}$$

Then, we can easily show that $\lim_n \|[g_n(a), b]\|=0$ for $a, b\in\mathfrak{L}$. Clearly, the trace ψ on $\mathfrak{L}$ is G-invariant—that is, $\psi(g(a))=\psi(a)$ for $g\in G$ and $a\in\mathfrak{L}$.

Let $\{\Pi_\psi, \mathfrak{H}_\psi\}$ be the *-representation of $\mathfrak{L}$ on a Hilbert space $\mathfrak{H}_\psi$ constructed via ψ, then there exists a unitary representation $g\to U_g$ of G on $\mathfrak{H}_\psi$ such that $U_g 1_\psi=1_\psi$ for $g\in G$ and $\Pi_\psi(g(a))=U_g\Pi_\psi(a)U_g{}^*$ for $a\in\mathfrak{L}$, where 1_ψ is the image of 1 in $\mathfrak{H}_\psi$. Let $\mathfrak{M}$ be the weak closure of $\Pi_\psi(\mathfrak{L})$ on $\mathfrak{H}_\psi$, then $\mathfrak{M}$ is a finite factor (cf. [3]). The mapping $x\to U_g x U_g{}^*(x\in\mathfrak{M})$ will define a *-automorphism ρ_g on $\mathfrak{M}$.

Definition 2. The finite factor $\mathfrak{M}$ is called the canonical infinite W^*-tensor product of finite factors $\{\mathfrak{A}_n\}$ and denoted by $\bigotimes_{n=1}^{\infty}\mathfrak{A}_n$.

Proposition 1. $\bigotimes_{n=1}^{\infty}\mathfrak{A}_n$ is asymptotically abelian.

Proof. Let τ be the normalized trace on $\bigotimes_{n=1}^{\infty}\mathfrak{A}_n$. We shall identify $\mathfrak{L}$ with the image $\Pi_\psi(\mathfrak{L})$. Then, $\tau=\psi$ on $\mathfrak{L}$. Let $x, y\in\bigotimes_{n=1}^{\infty}\mathfrak{A}_n$, then by Kaplansky's density theorem, there exist two sequences (x_m) and (y_m) in $\mathfrak{L}$ such that $\|x_m\|\leqq\|x\|$, $\|y_m\|\leqq\|y\|$ and $\|x_m-x\|_2\to 0$, $\|y_m-y\|_2\to 0$ $(m\to\infty)$, where $\|v\|_2=\tau(v^*v)^{1/2}$ for $v\in\bigotimes_{n=1}^{\infty}\mathfrak{A}_n$.

Then

$$\begin{aligned}
&\|[\rho g_n(x), y]-[\rho g_n(x_m), y_m]\|_2 \\
&\quad\leqq\|[\rho g_n(x), y]-[\rho g_n(x), y_m]+[\rho g_n(x), y_m]-[\rho g_n(x_m), y_m]\|_2 \\
&\quad\leqq\|[\rho g_n(x), y-y_m]\|_2+\|[\rho g_n(x)-\rho g_n(x_m), y_m]\|_2 \\
&\quad\leqq\|\rho g_n(x)(y-y_m)\|_2+\|(y-y_m)\rho g_n(x)\|_2+\|\rho g_n(x-x_m)y_m\|_2 \\
&\qquad+\|y_m\rho g_n(x-x_m)\|_2 \\
&\quad\leqq\|\rho g_n(x)\|\,\|y-y_m\|_2+\|y-y_m\|_2\|\rho g_n(x)\|+\|\rho g_n(x-x_m)\|_2\|y_m\| \\
&\qquad+\|y_m\|\|\rho g_n(x-x_m)\|_2 \\
&\quad=2\|x\|\,\|y-y_m\|_2+2\|y_m\|\|x-x_m\|_2\to 0 \qquad (m\to\infty)\,.
\end{aligned}$$

Hence, for arbitrary $\varepsilon>0$, there exists an m_0 such that

$$| \, ||[\rho g_n(x), y]||_2 - ||[\rho g_n(x_{m_0}), y_{m_0}]||_2 | < \varepsilon \qquad \text{for all } n\,.$$

On the other hand, $||[\rho g_n(x_{m_0}), y_{m_0}]|| < \varepsilon$ for $n \geqq n_0$, where n_0 is some integer; hence $||[\rho g_n(x), y]||_2 < 2\varepsilon$ for $n \geqq n_0$. This completes the proof.

Now let Φ be a countable discrete group, and let $\mathfrak{L}(\Phi)$ be the W^*-algebra generated by the left regular representation of Φ. We show examples of asymptotically abelian finite factors.

Example 1. Let $\mathfrak{L}_1$ be the type I_1-factor, then clearly it is asymptotically abelian.

Example 2. Let Π be the countable discrete group of all finite permutations on the set of all positive integers, then the W^*-algebra $\mathfrak{L}(\Pi)$ is a hyperfinite II_1-factor (cf. [6]). Since all hyperfinite II_1-factors on separable Hilbert spaces are $*$-isomorphic (cf. [6]), $\mathfrak{L}(\Pi)$ is $*$-isomorphic to $\bigotimes_{n=1}^{\infty} \mathfrak{L}_{2,n}$, with $\mathfrak{L}_{2,n} = \mathfrak{L}_2$, where $\mathfrak{L}_2$ is the type I_2-factor. Since the asymptotic abelianness is preserved under a $*$-isomorphism, by Proposition 1, $\mathfrak{L}(\Pi)$ is asymptotically abelian.

Example 3. Let Φ_2 be the countable discrete, free group with two generators, then the W^*-algebra $\mathfrak{L}(\Phi_2)$ is a II_1-factor (cf. [6]). Let $\bigotimes_{n=1}^{\infty} \mathcal{D}_n$ with $\mathcal{D}_n = \mathfrak{L}(\Phi_2)$, then by Proposition 1, $\bigotimes_{n=1}^{\infty} \mathcal{D}_n$ is asymptotically abelian.

Now, we shall show examples of finite factors which are not asymptotically abelian.

Example 4. Let $\mathfrak{L}_p$ be the type I_p-factor with $2 \leqq p < +\infty$ (p integer), then $\mathfrak{L}_p$ is not asymptotically abelian.

Proof. Let (ρ_n) be a sequence of $*$-automorphisms on $\mathfrak{L}_p$. Let $B(\mathfrak{L}_p)$ be the Banach algebra of all bounded operators on $\mathfrak{L}_p$, then $B(\mathfrak{L}_p)$ is finite-dimensional; therefore there exists a subsequence (ρ_{n_j}) of (ρ_n) such that $||\rho_{n_j} - T|| \to 0$ $(j \to \infty)$, where T is a bounded operator on $\mathfrak{L}_p$. It is easy to show that T is also a $*$-automorphism on $\mathfrak{L}_p$; hence clearly $\mathfrak{L}_p$ is not asymptotically abelian.

Example 5. Let Φ_2 be the countable discrete, free group with two generators, then $\mathfrak{L}(\Phi_2)$ is not asymptotically abelian.

Proof. Suppose that $\mathfrak{L}(\Phi_2)$ is asymptotically abelian, and let (ρ_n) be a family of *-automorphisms such that $\|[\rho_n(a), b]\|_2 \to 0$ $(n \to \infty)$ for $a, b \in \mathfrak{L}(\Phi_2)$.

Clearly, there exists a unitary element u in $\mathfrak{L}(\Phi_2)$ such that $\tau(u)=0$, where τ is the normalized trace on $\mathfrak{L}(\Phi_2)$. Then $\|[\rho_n(u), b]\|_2 = \|\rho_n(u)b - b\rho_n(u)\|_2 = \|\rho_n(u)b\rho_n(u)^* - b\|_2 \to 0$ $(n \to \infty)$ for $b \in \mathfrak{L}(\Phi_2)$.

Since $\rho_n(u)$ is unitary and $\tau(\rho_n(u)) = \tau(u) = 0$, $\mathfrak{L}(\Phi_2)$ has the property Γ; this is a contradiction (cf. [6]).

Example 6. Let Π be the group of all finite permutations on the set of all positive integers, and let Φ_2 be the free group of two generators, and let $\Phi_2 \times \Pi$ be the direct product group of Φ_2 and Π. Then, $\mathfrak{L}(\Phi_2 \times \Pi)$ is *-isomorphic to the W^*-tensor product $\mathfrak{L}(\Phi_2) \overline{\otimes} \mathfrak{L}(\Pi)$ of $\mathfrak{L}(\Phi_2)$ and $\mathfrak{L}(\Pi)$ (cf. [6], [9]).

In the following considerations, we shall show that $\mathfrak{L}(\Phi_2 \times \Pi)$ is not asymptotically abelian.

Lemma 1. Let Φ be a group and let E be a subset of Φ. Suppose there exist a subset $F \subset E$ and two elements $g_1, g_2 \in \Phi$ such that (i) $F \cup g_1 F g_1^{-1} = E$; (ii) F, $g_2^{-1} F g_2$ and $g_2 F g_2^{-1}$ are contained in E and mutually disjoint. Let $f(g)$ be a complex valued function on Φ such that $\sum_{g \in \Phi} |f(g)|^2 < +\infty$ and $(\sum_{g \in \Phi} |f(g_i g g_i^{-1}) - f(g)|^2)^{1/2} < \varepsilon$ $(i=1, 2)$. Then, $(\sum_{g \in E} |f(g)|^2)^{1/2} < K\varepsilon$, where K does not depend on ε and f.

Proof. $\nu(x) = \sum_{g \in X} |f(g)|^2$ for a subset $X \subset \Phi$. Then,

$$\varepsilon > (\sum_{g \in \Phi} |f(g_1 g g_1^{-1}) - f(g)|^2)^{1/2} \geqq |\nu(g_1 F g_1^{-1})^{1/2} - \nu(F)^{1/2}|.$$

Putting $\nu(E)^{1/2} = s$, then

$$|\nu(g_1 F g_1^{-1}) - \nu(F)| = |\nu(g_1 F g_1^{-1})^{1/2} + \nu(F)^{1/2}| \cdot |\nu(g_1 F g_1^{-1})^{1/2} - \nu(F)^{1/2}|$$
$$< 2s\varepsilon \quad \text{and so} \quad \nu(g_1 F g_1^{-1}) < \nu(F) + 2s\varepsilon\,;$$

hence

$$s^2 \leqq \nu(g_1 F g_1^{-1}) + \nu(F) < 2(\nu(F) + s\varepsilon)\,,$$

so that

$$\nu(F)>\frac{s^2}{2}-s\varepsilon .$$

Since

$$\Big(\sum_{g\in\Phi}|f(g_2gg_2^{-1})-f(g)|^2\Big)^{1/2}=\Big(\sum_{g\in\Phi}|f(g_2g_2^{-1}gg_2g_2^{-1})-f(g_2^{-1}gg_2)|^2\Big)^{1/2},$$

analogously we have

$$|\nu(g_2Fg_2^{-1})-\nu(F)|<2s\varepsilon$$

and

$$|\nu(g_2^{-1}Fg_2)-\nu(F)|<2s\varepsilon ;$$

hence

$$\nu(g_2Fg_2^{-1})>\nu(F)-2s\varepsilon>\frac{s^2}{2}-3s\varepsilon$$

and

$$\nu(g_2^{-1}Fg_2)>\frac{s^2}{2}-3s\varepsilon .$$

Therefore,

$$s^2 = \nu(E)\geqq\nu(F)+\nu(g_2^{-1}Fg_2)+\nu(g_2Fg_2^{-1})>\frac{3}{2}s^2-7s\varepsilon ;$$

hence

$$s<14\varepsilon .$$

This completes the proof.

Now, let us consider the group $\Phi_2\times\Pi$. Let k_1, k_2 be the generators of the group Φ_2, and let F_1 be the set of elements $\in\Phi_2$ which, when written as a power of k_1, k_2 of minimum length, end with k_1^n, $n=\pm1, \pm2, \cdots$. Let $F=F_1\times\Pi$ and let $a_1=(k_1, e)$ and $a_2=(k_2, e)$, where e is the unit of Π.

Then,

$$a_1Fa_1^{-1} = (k_1F_1k_1^{-1}, \Pi)$$

and

$$a_2Fa_2^{-1} = (k_2F_1k_2^{-1}, \Pi) ;$$

moreover

$$F\cup a_1Fa_1^{-1} = (F_1, \Pi)\cup(k_1F_1k_1^{-1}, \Pi) = (F_1\cup k_1Fk_1^{-1}, \Pi) = (e, \Pi)^c ,$$

where $(\cdot)^c$ is the complement of $(\cdot)$; F, $a_2^{-1}Fa_2$ and $a_2Fa_2^{-1}$ are contained in $(e, \Pi)^c$ and mutually disjoint. Hence by Lemma 1, we have

Lemma 2. Suppose that (f_n) be a sequence of complex valued functions on $\Phi_2\times\Pi$ such that $(\sum_{a\in\Phi_2\times\Pi}|f_n(a)|^2)^{1/2}<+\infty$ and

$$\lim_{n\to\infty}(\sum_{a\in\Phi_2\times\Pi}|f_n(a_iaa_i^{-1})-f_n(a)|^2)^{1/2}=0 \qquad (i=1, 2).$$

Then,

$$\lim_{n\to\infty}(\sum_{a\in(e,\Pi)^c}|f_n(a)|^2)^{1/2}=0.$$

Now we shall show

Theorem 1. $\mathfrak{L}(\Phi_2\times\Pi)$ is not asymptotically abelian.

Proof. Suppose that $\mathfrak{L}(\Phi_2=\Pi)$ is asymptotically abelian, and let (ρ_n) be a sequence of *-automorphisms on $\mathfrak{L}(\Phi_2\times\Pi)$ such that

$$||[\rho_n(x), y]||_2\to 0 \qquad (n\to\infty)$$

for $x, y\in\mathfrak{L}(\Phi_2\times\Pi)$.

Let $\varepsilon_t(t\in\Phi_2\times\Pi)$ be the unitary element of $\mathfrak{L}(\Phi_2\times\Pi)$ such that $(\varepsilon_t f)(a)=f(t^{-1}a)$ for $f\in\mathfrak{l}^2(\Phi_2\times\Pi)$ and $a\in\Phi_2\times\Pi$, where $\mathfrak{l}^2(\Phi_2\times\Pi)$ is the Hilbert space of all complex valued square summable functions on $\Phi_2\times\Pi$.

Since all elements of $\mathfrak{L}(\Phi_2\times\Pi)$ are realized as left convolution operators by elements of a subset of $\mathfrak{l}^2(\Phi_2\times\Pi)$ (cf. [6]), we shall embed $\mathfrak{L}(\Phi_2\times\Pi)$ into $\mathfrak{l}^2(\Phi_2\times\Pi)$. Then, $x\in\mathfrak{L}(\Phi_2\times\Pi)$ is a complex valued square summable function on $\Phi_2\times\Pi$.

Now let $x_1, x_2, \cdots, x_p$ be a finite subset of elements of $\mathfrak{L}(\Phi_2\times\Pi)$. Then,

$$||[\rho_n(x_j), \varepsilon_{a_i}]||_2\to 0 \qquad (n\to\infty)$$

for $i=1, 2$ and $j=1, 2, \cdots, p$.

$$\begin{aligned}||[\rho_n(x_j), \varepsilon_{a_i}]||_2 &= ||\rho_n(x_j)\varepsilon_{a_i}-\varepsilon_{a_i}\rho_n(x_j)||_2\\ &= ||\varepsilon_{a_i}^{-1}\rho_n(x_j)\varepsilon_{a_i}-\rho_n(x_j)||_2=(\sum_{a\in\Phi_2\times\Pi}|\rho_n(x_j)(a_iaa_i^{-1})-\rho_n(x_j)(a)|^2)^{1/2}\end{aligned}$$

Hence, by Lemma 2,

$$(\sum_{a\in(e,\Pi)^c}|\rho_n(x_j)(a)|^2)^{1/2}\to 0 \qquad (n\to\infty).$$

Put

$$f_n(x_j)(a)=\begin{cases}\rho_n(x_j)(a) & \text{if } a\in(e,\Pi)\\ 0 & \text{if } a\notin(e,\Pi)\end{cases}$$

then we can easily show that $f_n(x_j)\in\mathfrak{L}(\Phi_2\times\Pi)$. Let $\mathfrak{N}=\{\mathfrak{l}\mid\mathfrak{l}(a)=0$ for $a\notin(e,\Pi)$ and $\mathfrak{l}\in\mathfrak{L}(\Phi_2\times\Pi)\}$, then $\mathfrak{N}$ is a W^*-subalgebra of $\mathfrak{L}(\Phi_2\times\Pi)$; moreover put $\tilde{\mathfrak{l}}(h)=\mathfrak{l}(e,h)$ for $h\in\Pi$ and $\mathfrak{l}\in\mathfrak{N}$, then the mapping $\mathfrak{l}\to\tilde{\mathfrak{l}}$ is a $*$-isomorphism of $\mathfrak{N}$ onto the II_1-factor $\mathfrak{L}(\Pi)$; hence $\mathfrak{N}$ is a hyperfinite II_1-factor.

For arbitrary $\varepsilon>0$, there exists a positive integer n_0 such that

$$\Big(\sum_{a\in(e,\Pi)^c}(\rho_{n_0}(x_j)(a)\,|^2)^{1/2}<\varepsilon \qquad \text{for} \quad j=1,2,\cdots,p\,.$$

Then,

$$\|\rho_{n_0}(x_j)-f_{n_0}(x_j)\|_2<\varepsilon \qquad \text{for} \quad j=1,2,\cdots,p\,.$$

Since $\mathfrak{N}$ is a hyperfinite II_1-factor, there exist a type I_{n_p} subfactor $\mathfrak{L}_{n_p}$ of $\mathfrak{N}$ and elements $r_1, r_2, \cdots, r_n\in\mathfrak{L}_{n_p}$ such that

$$\|f_{n_0}(x_j)-r_j\|_2<\varepsilon \qquad \text{for} \quad j=1,2,\cdots,p\,.$$

Therefore,

$$\|\rho_{n_0}(x_j)-r_j\|_2<2\varepsilon \qquad \text{for} \quad j=1,2,\cdots,p\,.$$

Since ρ_{n_0} is a $*$-automorphism, $\|x_j-\rho_{n_0}^{-1}(r_j)\|_2<2\varepsilon$ and $\rho_{n_0}^{-1}(r_j)\in\rho_{n_0}^{-1}(\mathfrak{L}_{n_p})$ for $j=1,2,\cdots,p$. $\rho_{n_0}^{-1}(\mathfrak{L}_{n_p})$ is a type I_{n_p} factor and $\mathfrak{L}(\Phi_2\times\Pi)$ is a II_1-factor on a separable Hilbert space; hence by the result of Murray and von Neumann [6], $\mathfrak{L}(\Phi_2\times\Pi)$ is a hyperfinite II_1-factor.

On the other hand, by Schwartz's theorem [10], $\mathfrak{L}(\Phi_2\times\Pi)$ is not hyperfinite. This is a contradication and completes the proof.

Now we shall show the existence of the fifth example of II_1-factors on separable Hilbert spaces.

Corollary 1. $\mathfrak{L}(\Phi_2\times\Pi)$ is not $*$-isomorphic to $\bigotimes_{n=1}^{\infty}\mathcal{D}_n$ with $\mathcal{D}_n=\mathfrak{L}(\Phi_2)$.

Proof. Clearly the asymptotic abelianness is preserved under $*$-isomorphisms; hence $\mathfrak{L}(\Phi_2\times\Pi)$ is not $*$-isomorphic to $\bigotimes_{n=1}^{\infty}\mathcal{D}_n$. This completes the proof.

Proposition 2. $\bigotimes_{n=1}^{\infty}\mathcal{D}_n\overline{\otimes}\mathfrak{L}(\Pi)=\bigotimes_{n=1}^{\infty}\mathcal{D}_n$, where $\mathcal{D}_n=\mathfrak{L}(\Phi_2)$ and $(\cdot)\overline{\otimes}(\cdot\cdot)$ is the W^*-tensor product of $(\cdot)$ and $(\cdot\cdot)$.

Proof. Since $\mathfrak{L}(\Phi_2)$ is a II_1-factor, there exists a type I_2-factor $\mathfrak{L}_2$ such that $\mathfrak{L}(\Phi_2)=\mathfrak{L}_2\otimes\mathfrak{L}_2'$, where $\mathfrak{L}_2'$ is the commutant of $\mathfrak{L}_2$ in $\mathfrak{L}(\Phi_2)$; hence $\bigotimes_{n=1}^{\infty}\mathcal{D}_n=\bigotimes_{n=1}^{\infty}\mathfrak{L}_{2,n}\otimes\mathfrak{L}_{2',n}'$, where $\mathfrak{L}_{2,n}=\mathfrak{L}_2$ and $\mathfrak{L}_{2',n}'=\mathfrak{L}_2'$.

Hence

$$\bigotimes_{n=1}^{\infty}\mathcal{D}_n\otimes\mathfrak{L}(\Pi) = \left(\bigotimes_{n=1}^{\infty}\mathfrak{L}_{2,n}'\right)\overline{\otimes}\bigotimes_{n=1}^{\infty}\mathfrak{L}_{2,n}\overline{\otimes}\mathfrak{L}(\Pi)$$
$$= \bigotimes_{n=1}^{\infty}\mathfrak{L}_{2,n}'\otimes\mathfrak{L}_{2\ n} = \bigotimes_{n=1}^{\infty}\mathcal{D}_n\,,$$

because $\bigotimes_{n=1}^{\infty}\mathfrak{L}_{2,n}$ and $\mathfrak{L}(\Pi)$ are hyperfinite and so $\bigotimes_{n=1}^{\infty}\mathfrak{L}_{2,n}\overline{\otimes}\mathfrak{L}(\Pi)$ is also hyperfinite.

This completes the proof.

The following defintion is due to Ching [1].

Definition 3. A finite factor M is said to have property C, if for each sequence $u_n(n=1, 2, \cdots)$ of unitary elements in M with the property that

$$||u_n{}^*xu_n-x||_2\to 0 \qquad (n\to\infty)$$

for each $x\in M$, there exists a uniformly bounded sequence v_n $(n=1, 2, \cdots)$ of mutually commuting elements in M such that $||u_n-v_n||_2\to 0$ $(n\to\infty)$.

Then, Ching [1] proved that $\mathfrak{L}(\Pi)$ and $\mathfrak{L}(\Phi_2\times\Pi)$ do not have property C and also there exists a type II_1-factor M_4 which has both of properties C and Γ.

It is not so difficult to see that $\mathfrak{L}(\Phi_2)$ has property C, although we do not need it here.

Corollary 2. $\bigotimes_{n=1}^{\infty}\mathcal{D}_n$ with $\mathcal{D}_n=\mathfrak{L}(\Phi_2)$ does not have property C.

Let g_i be the element in Π which permutes i and $i+1$ and leaves all other positive integers fixed, for each $i=1, 2, \cdots$.

Clearly $||\varepsilon_{g_i}{}^*x\varepsilon_{g_i}{}^{-1}-x||_2\to 0$ for $x\in\mathfrak{L}(\Pi)$. Hence let 1 be the unit of $\bigotimes_{n=1}^{\infty}\mathcal{D}_n$, then $||1\otimes\varepsilon^*{}_{g_i}y1\otimes\varepsilon_{g_i}-y||_2\to 0$ for $y\in\bigotimes_{n=1}^{\infty}\mathcal{D}_n\overline{\otimes}\mathfrak{L}(\Pi)$. Suppose $\bigotimes_{n=1}^{\infty}\mathcal{D}_n$ has property C, then $\bigotimes_{n=1}^{\infty}\mathcal{D}_n\overline{\otimes}\mathfrak{L}(\Pi)$ has property C.

Let $\{v_i\}$ be a uniformly bounded sequence of mutually commuting elements in $\bigotimes_{n=1}^{\infty}\mathcal{D}_n\overline{\otimes}\mathfrak{L}(\Pi)$ such that $||1\otimes\varepsilon_{g_i}-v_i||_2\to 0$ $(i\to\infty)$. Then, since $g_ig_{i+1}\neq g_{i+1}g_i$,

$$\begin{aligned}\sqrt{2} &= \|1\otimes\varepsilon_{(g_i g_{i+1})} - 1\otimes\varepsilon_{(g_{i+1} g_i)}\|_2 \\ &= \|1\otimes\varepsilon_{g_i} 1\otimes\varepsilon_{g_{i+1}} - 1\otimes\varepsilon_{g_{i+1}} 1\otimes\varepsilon_{g_i}\|_2 \\ &\leqq \|(1\otimes\varepsilon_{g_i} - v_i) 1\otimes\varepsilon_{g_{i+1}}\|_2 + \|v_i(1\otimes\varepsilon_{g_{i+1}} - v_{i+1})\|_2 \\ &\quad + \|(v_{i+1} - 1\otimes\varepsilon_{g_{i+1}})v_i\|_2 + \|1\otimes\varepsilon_{i+1}(v_i - 1\otimes\varepsilon_{g_i})\|_2 \to 0 \qquad (n\to\infty).\end{aligned}$$

This is a contradiction and completes the proof.

Proposition 3. There are five examples of II_1–factors with different algebraical types on separable Hilbert spaces.

Proof. $\bigotimes_{n=1}^{\infty} \mathscr{D}_n$ with $\mathscr{D}_n = \mathfrak{L}(\Phi_2) \oplus \mathfrak{L}(\Pi)$, because $\mathfrak{L}(\Pi)$ can not contain the II_1–factor which is *–isomorphic to $\mathfrak{L}(\Phi_2)$ as W^*–subalgebra (cf. [10]).

Clearly $\bigotimes_{n=1}^{\infty} \mathscr{D}_n \neq \mathfrak{L}(\Phi_2)$, because $\bigotimes_{n=1}^{\infty} \mathscr{D}_n = \bigotimes_{n=1}^{\infty} \mathscr{D}_n \otimes \mathfrak{L}(\Pi)$ has property Γ; by Theorem 1 $\bigotimes_{n=1}^{\infty} \mathscr{D}_n \neq \mathfrak{L}(\Phi_2 \times \Pi)$; by Corollary 2, $\bigotimes_{n=1}^{\infty} \mathscr{D}_n \neq M_4$.

This completes the proof.

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