

Calcul de certaines valeurs de $N(E)$ et de $N_{\text{osc}}(E)$ pour les 2 001 052 premiers zéros de zeta fournis par Odlyzko

On souhaite calculer les valeurs de $N(E)$ et de $N_{\text{osc}}(E)$ apparaissant dans les formules de l'article [1]. On calcule les valeurs de $N_{\text{osc}}(E)$ et de $N(E)$ en omettant le $N_{\text{osc}}(E)$ qui est minuscule rapporté à $N(E)$ et la constante $o(1)$ pour x variant de 1000 en 1000 jusqu'à 2 millions.

$$N(E) = \# \text{ de zéros } \rho, \quad 0 < \text{Im } \rho < E,$$

$$N(E) = \frac{E}{2\pi} \left(\log \left(\frac{E}{2\pi} \right) - 1 \right) + \frac{7}{8} + o(1) + N_{\text{osc}}(E),$$

$$N_{\text{osc}}(E) = \frac{1}{\pi} \text{Im} \log \zeta \left(\frac{1}{2} + iE \right).$$

On utilise le fichier des premiers zéros de zeta fournis par Odlyzko à cette page https://www-users.cse.umn.edu/~odlyzko/zeta_tables/index.html (dernier fichier de 14 méga des 2 millions et quelques premiers zéros).

Voici le programme python.

```
import mpmath
from mpmath import zeta, log, pi

f = open('zero', 'r') ; zeros = []
for line in f.readlines():
    zeros.append(float(line))
f.close() ; print(zeros) ; listeetapes = []
for k in range(1,202):
    listeetapes.append(1000*k) ; print(listeetapes)

LP1 = zeros ; print('j ai ',len(zeros), ' zeros.')
LP2 = listeetapes ; print('j ai ',len(listeetapes), ' etapes de 1000 en 1000.')
listeN = [] ; nb = 0 ; indice = 0
for val in LP2:
    while LP1[indice] < val:
        nb = nb+1
        indice = indice+1
    listeN.append(nb+1)
print(listeN)
listeNosc = []
for val in LP2:
    Noscval = (1/pi)*log(zeta(0.5+1j*val)).imag
    print('Nosc(',val, ') = ',Noscval)
    listeNosc.append(Noscval)
listeN = []
for val in LP2:
    Nval = (val/(2*pi))*(log(val/(2*pi))-1)+7/8
    print('N(',val, ') = ',Nval)
    listeN.append(Nval)
```

Voici son résultat.

```
C:~2022>python3 pgm-trace-AC.py
```

```
j ai 2001052 zeros.
```

```
j ai 201 etapes de 1000 en 1000.
```

```
[650, 1518, 2470, 3475, 4521, 5599, 6704, 7831, 8979, 10143, 11325, 12520, 13729,
14951, 16183, 17426, 18679, 19942, 21212, 22492, 23780, 25075, 26377, 27687, 29003,
30325, 31654, 32989, 34328, 35674, 37025, 38380, 39741, 41108, 42477, 43852, 45232,
46615, 48004, 49396, 50792, 52191, 53595, 55003, 56413, 57828, 59246, 60668, 62092,
63520, 64951, 66385, 67823, 69263, 70706, 72153, 73601, 75053, 76508, 77964, 79425,
80887, 82352, 83819, 85289, 86762, 88237, 89714, 91194, 92675, 94160, 95646, 97134,
98626, 100119, 101613, 103111, 104611, 106111, 107615, 109121, 110627, 112136, 113648,
115161, 116676, 118193, 119711, 121232, 122753, 124278, 125803, 127331, 128860,
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161338, 162901, 164464, 166030, 167597, 169165, 170735, 172306, 173877, 175451,
177026, 178601, 180179, 181758, 183337, 184918, 186501, 188084, 189669, 191254,
192842, 194429, 196018, 197610, 199201, 200794, 202389, 203983, 205579, 207176,
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273541, 275178, 276817, 278458, 280098, 281740, 283382, 285025, 286669, 288315,
289961, 291606, 293254, 294902, 296551, 298201, 299852]
```

Valeurs de $N_{osc}(E)$

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Nosc( 2000 ) = 0.00692590568567653
Nosc( 3000 ) = 0.352563001522947
Nosc( 4000 ) = -0.382343520339771
Nosc( 5000 ) = -0.331174708068316
Nosc( 6000 ) = -0.326671139180878
Nosc( 7000 ) = 0.0278186179979788
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Nosc( 22000 ) = -0.185263881968176
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Nosc(200000) = 0.198146222665143
Nosc(201000) = 0.65134299717344

Valeurs de N(E)

N(1000) = 648.616235312967
N(2000) = 1516.99307077859
N(3000) = 2468.64743478799
N(4000) = 3474.38234186248
N(5000) = 4520.33117338178
N(6000) = 5598.32667003394
N(7000) = 6702.97218043465
N(8000) = 7830.43208433556
N(9000) = 8977.83849091124
N(10000) = 10142.9653475268
N(11000) = 11324.0343309507
N(12000) = 12519.5919409838
N(13000) = 13728.4278563336
N(14000) = 14949.5185619379
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N(16000) = 17425.0739698923
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N(20000) = 22491.4116965801
N(21000) = 23779.0076432476
N(22000) = 25074.1852635805
N(23000) = 26376.5996927252
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N(25000) = 29001.9058509936
N(26000) = 30324.2435146516
N(27000) = 31652.7040323755
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N(54000) = 69261.6942688727
N(55000) = 70704.9216197398
N(56000) = 72151.0428562999
N(57000) = 73600.0062964791
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N(60000) = 77963.4596667222
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N(62000) = 80885.7695568402
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N(67000) = 88235.7779413603
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N(73000) = 97133.8766360128
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N(83000) = 112135.664544656
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N(90000) = 122752.611858687
N(91000) = 124276.555926505
N(92000) = 125802.248984945
N(93000) = 127329.672022484
N(94000) = 128858.806436476
N(95000) = 130389.634020099

N(96000) = 131922.136949852
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N(98000) = 134992.099398979
N(99000) = 136529.525082596
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N(130000) = 184917.217490499
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N(133000) = 189667.449844442
N(134000) = 191253.266092948
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N(137000) = 196017.80618196
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N(140000) = 200792.802540939
N(141000) = 202386.75259986
N(142000) = 203981.831426706
N(143000) = 205578.031072274
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N(146000) = 210373.277083169
N(147000) = 211973.882700855
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N(150000) = 216782.166559865

N(151000) = 218387.059454821
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 N(153000) = 221600.000352182
 N(154000) = 223208.034576429
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 N(156000) = 226427.196799969
 N(157000) = 228038.311549324
 N(158000) = 229650.440031286
 N(159000) = 231263.57582974
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 N(161000) = 234492.844113685
 N(162000) = 236108.964164469
 N(163000) = 237726.066659412
 N(164000) = 239344.145571174
 N(165000) = 240963.194945919
 N(166000) = 242583.208901983
 N(167000) = 244204.181628562
 N(168000) = 245826.107384449
 N(169000) = 247448.980496781
 N(170000) = 249072.795359835
 N(171000) = 250697.546433835
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 N(173000) = 253949.835378418
 N(174000) = 255577.362488927
 N(175000) = 257205.804288052
 N(176000) = 258835.155548944
 N(177000) = 260465.411104152
 N(178000) = 262096.565844612
 N(179000) = 263728.614718669
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 N(181000) = 266995.374942232
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 N(183000) = 270265.652473682
 N(184000) = 271902.098183933
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 N(186000) = 275177.579859164
 N(187000) = 276816.606523229
 N(188000) = 278456.484287302
 N(189000) = 280097.208624214
 N(190000) = 281738.7750547
 N(191000) = 283381.179146649
 N(192000) = 285024.416514359
 N(193000) = 286668.482817811
 N(194000) = 288313.373761964
 N(195000) = 289959.085096055
 N(196000) = 291605.612612919
 N(197000) = 293252.952148321
 N(198000) = 294901.099580305
 N(199000) = 296550.05082855
 N(200000) = 298199.801853744
 N(201000) = 299850.34865697

Référence

- [1] A. Connes, *Trace formula in noncommutative geometry and the zeros of the Riemann zeta function*. *Selecta Math.* (N.S.) 5 (1999), no. 1, 29–106.