Décomposition en valeurs singulières d'une matrice un peu creuse mais particulière (Denise Vella-Chemla, 13.3.2019)

On peut se reporter à cette première note ou à cette seconde note pour avoir une idée de ce que l'on tente de faire en ce moment : il s'agit de calculer la décomposition en valeurs singulières d'une matrice A choisie de façon un peu hasardeuse et d'étudier l'allure de la matrice intermédiaire Σ obtenue par la décomposition $A = U\Sigma V^*$.

La matrice A est une matrice triangulaire basse contenant des nombres complexes et définie par :

$$A[n,x] \;=\; \begin{cases} e^{\frac{2ix\pi}{n}} \text{ si } x \text{ et } n \text{ sont premiers}, 1 \leqslant x \leqslant n \,; \\ \\ 0 \text{ sinon.} \end{cases}$$

Voici les programmes : le programme en C++ écrit la matrice.

```
#include <iostream>
#include <stdio.h>
#include <cmath>
#include <complex>
using namespace std ;
#define M_PI 3.14159265358979323846
typedef std::complex<double> dcomplex ;
const dcomplex di = dcomplex(0.0,1.0) ;

int prime(int atester) {
  unsigned long diviseur=2;
  bool pastrouve=true;
  unsigned long k = 2;
  if (atester == 1) return 0;
  if (atester == 2) return 1;
  if (atester == 3) return 1;
  if (atester == 5) return 1;
  if (atester == 7) return 1;
  while (pastrouve) {
    if ((k * k) > atester) return 1;
    else
      if ((atester % k) == 0) {
        return 0 ;
      }
        else k++;
  }
}

int main (int argc, char* argv[]) {
  int n, x, nmax ;
  dcomplex mat[421][421] ;

  nmax = 100 ;
  for (n = 1 ; n <= nmax ; ++n)
    for (x = 1 ; x <= nmax ; ++x)
      mat[n][x] = 0 ;
  for (n = 1 ; n <= nmax ; ++n)
    if (prime(n))
      for (x = 1 ; x <= n ; ++x)
        if (prime(x))
          mat[n][x] = exp((dcomplex((float)x,0)*2.0*di*M_PI)/dcomplex((float)n,0.0)) ;
  for (n = 1 ; n <= nmax ; ++n) {
      std::cout << "[" ;
      for (x = 1 ; x <= nmax ; ++x)
        std::cout << mat[n][x] << "," ;
      std::cout << "]," ;
      std::cout << "\n" ;
  }
}
```

Des contraintes d'occupation mémoire font qu'on expérimente seulement sur une matrice de taille 100×100.

On rappelle que la matrice Σ contient sur sa diagonale les valeurs singulières de A, i.e. les racines carrées positives des valeurs propres de AA^* ou de A^*A (qui sont égales même si les produits matriciels AA^* et A^*A ne le sont pas forcément).

Les carrés des $\Sigma[x]$ sont les valeurs propres de AA^* ou de A^*A.

Le programme python ci-dessous décompose la matrice, obtenue par le programme C++, en valeurs singulières (Σ est remplacé par s dans le programme python) puis il calcule les valeurs des carrés des $Sigma[i]$:

```
# Reconstruct SVD
import numpy as np
from numpy import array
from numpy import diag
from numpy import dot
from numpy import zeros
from numpy.linalg import svd

A = array( # ici coller la matrice obtenue par le programme en C++ # )
print("A")
print(A)
U, s, V = np.linalg.svd(A)
print("\nU")
print(U)
print("\ns")
print(s)
print("\nV")
print(V)
print("\nA=UsV")
#Sigma = zeros((A.shape[0], A.shape[1]))
#Sigma[:A.shape[1], :A.shape[1]] = diag(s)
#print("\n On controle quon revient bien a la matrice initiale.")
#B = U.dot(Sigma.dot(V))
#print(B)
for i in range(100):
    print(s[i]**2)
    print( )
```

Voici le résultat de ce programme. On ne fournit que V et Σ, avec, en regard pour V, les indices entiers auxquels sont associés ces complexes.

```
V
[ [[ 1.00000000e+00  0.00000000e+00]..........................1
    [ 0.00000000e+00  1.00000000e+00]]
  [[-1.00000000e+00  2.44929000e-16]..........................2
    [ 2.44929000e-16  1.00000000e+00]]
  [[-8.66025606e-01 -4.99999650e-01]..........................3
    [-4.99999650e-01  8.66025606e-01]]
  [[ 1.00000000e+00  0.00000000e+00]..........................4
    [ 0.00000000e+00  1.00000000e+00]]
  [[-1.00000000e+00 -0.00000000e+00]..........................5
    [-0.00000000e+00 -1.00000000e+00]]
  [[ 1.00000000e+00  0.00000000e+00]..........................6
    [ 0.00000000e+00  1.00000000e+00]]
  [[ 6.23489736e-01 -7.81831535e-01]..........................7
    [ 7.81831535e-01  6.23489736e-01]]
  [[ 1.00000000e+00  0.00000000e+00]..........................8
    [ 0.00000000e+00  1.00000000e+00]]
  [[ 1.00000000e+00  0.00000000e+00]..........................9
    [ 0.00000000e+00  1.00000000e+00]]
  [[ 1.00000000e+00  0.00000000e+00].........................10
    [ 0.00000000e+00  1.00000000e+00]]
```

```
[[-7.38660966e-01 -6.74077130e-01]...........................11
  [ 6.74077130e-01 -7.38660966e-01]]
[[ 1.00000000e+00  0.00000000e+00]...........................12
  [ 0.00000000e+00  1.00000000e+00]]
[[-9.35016434e-01  3.54604382e-01]...........................13
  [-3.54604382e-01 -9.35016434e-01]]
[[ 1.00000000e+00  0.00000000e+00]...........................14
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................15
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................16
  [ 0.00000000e+00  1.00000000e+00]]
[[-3.78451830e-01 -9.25620988e-01]...........................17
  [ 9.25620988e-01 -3.78451830e-01]]
[[ 1.00000000e+00  0.00000000e+00]...........................18
  [ 0.00000000e+00  1.00000000e+00]]
[[-6.77282597e-01 -7.35722967e-01]...........................19
  [ 7.35722967e-01 -6.77282597e-01]]
[[ 1.00000000e+00  0.00000000e+00]...........................20
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................21
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................22
  [ 0.00000000e+00  1.00000000e+00]]
[[-6.50478929e-01 -7.59524300e-01]...........................23
  [ 7.59524300e-01 -6.50478929e-01]]
[[ 1.00000000e+00  0.00000000e+00]...........................24
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................25
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................26
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................27
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................28
  [ 0.00000000e+00  1.00000000e+00]]
[[-7.70779716e-01 -6.37101742e-01]...........................29
  [ 6.37101742e-01 -7.70779716e-01]]
[[ 1.00000000e+00  0.00000000e+00]...........................30
  [ 0.00000000e+00  1.00000000e+00]]
[[-9.01507120e-01 -4.32764270e-01]...........................31
  [ 4.32764270e-01 -9.01507120e-01]]
[[ 1.00000000e+00  0.00000000e+00]...........................32
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................33
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................34
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................35
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................36
  [ 0.00000000e+00  1.00000000e+00]]
[[-9.07897613e-01 -4.19191990e-01]...........................37
  [ 4.19191990e-01 -9.07897613e-01]]
[[ 1.00000000e+00  0.00000000e+00]...........................38
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................39
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................40
  [ 0.00000000e+00  1.00000000e+00]]
[[-8.98389049e-01 -4.39200544e-01]...........................41
  [ 4.39200544e-01 -8.98389049e-01]]
[[ 1.00000000e+00  0.00000000e+00]...........................42
  [ 0.00000000e+00  1.00000000e+00]]
[[-9.55128668e-01 -2.96191201e-01]...........................43
  [ 2.96191201e-01 -9.55128668e-01]]
[[ 1.00000000e+00  0.00000000e+00]...........................44
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................45
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................46
  [ 0.00000000e+00  1.00000000e+00]]
```

```
[[-9.67768909e-01 -2.51839908e-01]...........................47
  [ 2.51839908e-01 -9.67768909e-01]]
[[ 1.00000000e+00  0.00000000e+00]...........................48
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................49
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................50
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................51
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................52
  [ 0.00000000e+00  1.00000000e+00]]
[[-6.77877621e-01 -7.35174762e-01]...........................53
  [ 7.35174762e-01 -6.77877621e-01]]
[[ 1.00000000e+00  0.00000000e+00]...........................54
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................55
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................56
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................57
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................58
  [ 0.00000000e+00  1.00000000e+00]]
[[-7.02084591e-01 -7.12093552e-01]...........................59
  [ 7.12093552e-01 -7.02084591e-01]]
[[ 1.00000000e+00  0.00000000e+00]...........................60
  [ 0.00000000e+00  1.00000000e+00]]
[[-8.14806979e-01 -5.79732341e-01]...........................61
  [ 5.79732341e-01 -8.14806979e-01]]
[[ 1.00000000e+00  0.00000000e+00]...........................62
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................63
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................64
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................65
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................66
  [ 0.00000000e+00  1.00000000e+00]]
[[-8.51828819e-01 -5.23820259e-01]...........................67
  [ 5.23820259e-01 -8.51828819e-01]]
[[ 1.00000000e+00  0.00000000e+00]...........................68
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................69
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................70
  [ 0.00000000e+00  1.00000000e+00]]
[[-8.90461902e-01 -4.55057800e-01]...........................71
  [ 4.55057800e-01 -8.90461902e-01]]
[[ 1.00000000e+00  0.00000000e+00]...........................72
  [ 0.00000000e+00  1.00000000e+00]]
[[-9.33934364e-01 -3.57444546e-01]...........................73
  [ 3.57444546e-01 -9.33934364e-01]]
[[ 1.00000000e+00  0.00000000e+00]...........................74
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................75
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................76
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................77
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................78
  [ 0.00000000e+00  1.00000000e+00]]
[[-9.40182654e-01 -3.40670776e-01]...........................79
  [ 3.40670776e-01 -9.40182654e-01]]
[[ 1.00000000e+00  0.00000000e+00]...........................80
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................81
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................82
  [ 0.00000000e+00  1.00000000e+00]]
[[-9.42395225e-01 -3.34501481e-01]...........................83
  [ 3.34501481e-01 -9.42395225e-01]]
```

```
[[ 1.00000000e+00  0.00000000e+00]...........................84
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................85
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................86
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................87
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................88
  [ 0.00000000e+00  1.00000000e+00]]
[[-8.98788033e-01 -4.38383476e-01]...........................89
  [ 4.38383476e-01 -8.98788033e-01]]
[[ 1.00000000e+00  0.00000000e+00]...........................90
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................91
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................92
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................93
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................94
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................95
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................96
  [ 0.00000000e+00  1.00000000e+00]]
[[-8.07036946e-01 -5.90500946e-01]...........................97
  [ 5.90500946e-01 -8.07036946e-01]]
[[ 1.00000000e+00  0.00000000e+00]...........................98
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]...........................99
  [ 0.00000000e+00  1.00000000e+00]]
[[ 1.00000000e+00  0.00000000e+00]..........................100
  [ 0.00000000e+00  1.00000000e+00]]]

A=UsV
```

On est étonné par l'image du nombre premier 5, qui vaut $-1-i$.

La matrice Σ contient les nombres complexes suivants pour les indices de 1 à 100. Seuls les nombres premiers ont une image puisque c'est ce qui avait été choisi dans l'opérateur A initial :

0	0	0	0	2.6826396	2.40903408	3.21129004	2.7726549	0	0
1	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0
1.22474473	0.70710653	2.28586372	1.94289118	2.79906634	2.48298778	0	0	3.55554093	3.2184044
0	0	0	0	0	0	0	0	0	0
1.519545	0.83125352	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
1.54950625	1.26452807	0	0	2.83967352	2.63367708	3.29123225	2.85793428	0	0
0	0	0	0	0	0	0	0	0	0
0	0	2.48973762	1.94966722	0	0	0	0	3.60924318	3.31260671
0	0	0	0	0	0	0	0	0	0
1.72149134	1.42704872	2.60157616	2.05713421	0	0	3.3744563	2.93479958	0	0
0	0	0	0	0	0	0	0	0	0
1.79378393	1.66803534	0	0	2.93039612	2.72264253	3.46912883	2.99418543	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
2.07867357	1.63680052	2.58302332	2.30824337	0	0	0	0	3.71177469	3.35003389
0	0	0	0	0	0	0	0	0	0
2.1600962	1.82592019	0	0	3.10887717	2.70829899	3.4907486	3.13283731	0	0
0	0	0	0	0	0	0	0	0	0

Les images par $f(x) = \Sigma[x]^2$ des nombres de 1 à 100 sont les nombres complexes suivants (lire le tableau par colonnes) :

0	0	0	0	7.19655524	5.80344518	10.3123837	7.6876152	0	0
1	0	0	0	0	0	0	0	0	0
1.49999965	0.49999965	5.22517294	3.77482612	7.83477236	6.16522832	0	0	12.64187133	10.35812691
0	0	0	0	0	0	0	0	0	0
2.30901701	0.69098241	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
2.40096962	1.59903125	0	0	8.0637457	6.93625495	10.83220972	8.16778837	0	0
0	0	0	0	0	0	0	0	0	0
0	0	6.19879343	3.80120228	0	0	0	0	13.02663633	10.9733632
0	0	0	0	0	0	0	0	0	0
2.96353242	2.03646804	6.76819854	4.23180114	0	0	11.3869553	8.61304856	0	0
0	0	0	0	0	0	0	0	0	0
3.2176608	2.78234189	0	0	8.58722143	7.41278233	12.03485482	8.96514636	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
4.32088383	2.67911596	6.67200946	5.32798744	0	0	0	0	13.77727135	11.22272707
0	0	0	0	0	0	0	0	0	0
4.66601558	3.33398455	0	0	9.66511729	7.33488344	12.1853258	9.81466959	0	0
0	0	0	0	0	0	0	0	0	0