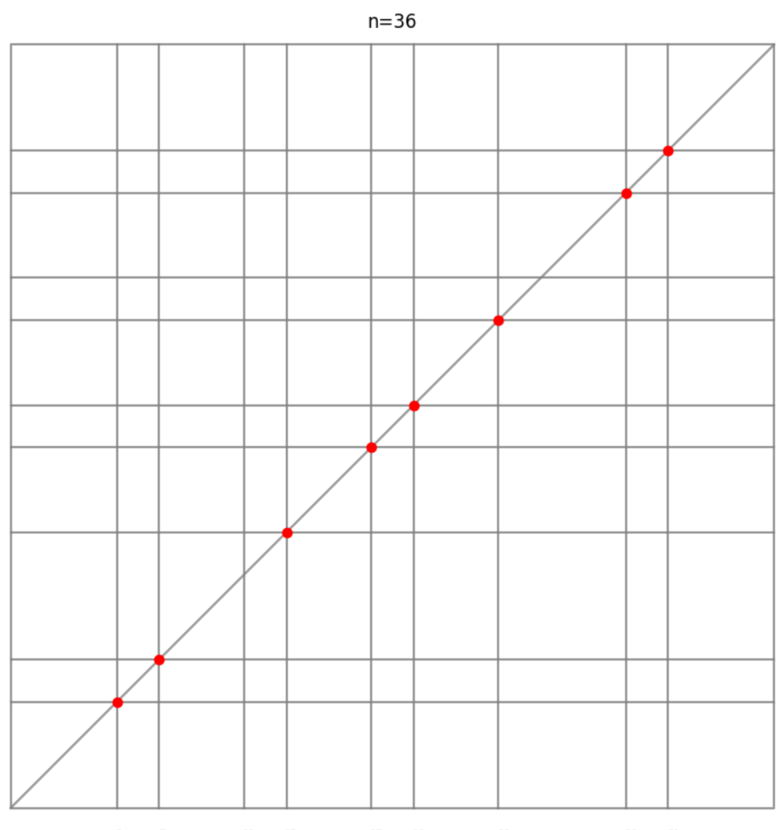
Scottish squares and fixed points in the complex plane (Denise Vella-Chemla, August 7, 2025)

1. Presentation of the problem

Still looking for a justification of Goldbach's conjecture, we reflect on how to visualize Goldbach's components in the complex plane.

We use a square with side of length n. On the x-axis are positioned the odd prime numbers p_k between 3 and $n-3$ inclusive and which do not divide n (we denote the set of these prime numbers $\mathcal{P}_n$).

We draw a network of lines whose vertical lines have the equations $x = p_k$ for each of the prime numbers of $\mathcal{P}_n$. We also draw horizontal lines $y = n - p_k$; This choice means that if we want to enumerate the prime numbers of $\mathcal{P}_n$ along the coordinate axes, we must do so as usual from left to right on the x-axis, but from top to bottom on the y-axis (getting closer and closer to the origin). Let's give an example: for $n = 36$, the only prime number "missing" in $\mathcal{P}_n$, because it does not divide n, is 3. The other prime numbers between 3 and 33 inclusive are those in the list: $[5, 7, 11, 13, 17, 19, 23, 29, 31]$.



2. Goldbach components of n and symmetry with respect to the SW-NE diagonal (with equation $x = y$)

As a reminder, we recall that in the complex plane, the image of the point with affix z by a symmetry with respect to the line determined by the points a and b is $z' = \alpha\overline{z} + \beta$ with $\alpha = \dfrac{a-b}{\overline{a}-\overline{b}}$ and $\beta = \dfrac{b\overline{a} - a\overline{b}}{\overline{a}-\overline{b}}$.

The SW-NE diagonal is determined by the points $a = 0$ and $b = n + ni$.

For this diagonal $\alpha = \dfrac{-n - ni}{-n + ni}$, $\beta = 0$.

Let's calculate the images of the points in the lattice of points chosen by the central symmetry with respect to the SW-NE diagonal (ascending, with equation $y = x$). The calculation program is provided in Appendix. Any point of the form $p + qi$ is mapped to the point $q + pi$. The file of square drawings for the numbers n between 6 and 103 can be viewed at:

https://denisevellachemla.eu/touscarrespourdg.pdf.

The Goldbach decomposing elements of n are colored in red on the SW-NE diagonal.

Remark: Even numbers that are doubles of a prime number (such as $38 = 2 \times 19$, or $94 = 2 \times 47$) trivially satisfy Goldbach's conjecture; they are the sum of two identical prime numbers. Their square has a red dot at the intersection of its diagonals, i.e., right in the middle of the square.

By simple reasoning, we understand that if a Goldbach decomposing element exists, it is a fixed point of a double application of the symmetry with respect to the SW-NE diagonal (with equation $y = x$) and that its coordinates are therefore equal.

Let's now see the number of points in the lattice of points.

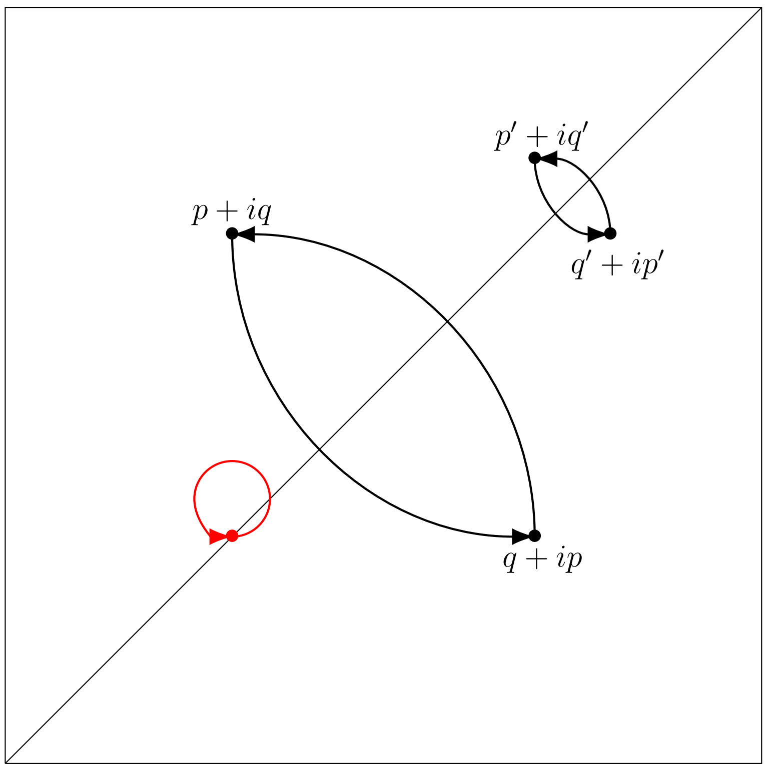
First case: if the number of prime numbers between 3 and $n-3$ (both inclusive) is odd; its square, which counts the number of points in the lattice, is also odd (indeed, $(2k+1)^2 = 4k^2 + 4k + 1 = (4k^2 + 4k) + 1 = 2k' + 1$), the function "symmetry with respect to the diagonal of the square" has an odd number of points as its domain. But since the fixed points of a squared symmetry (a symmetry with respect to a line is an idempotent application: its square is the identity) are on the line with respect to which this symmetry is carried out, if the lattice contains an odd number of points, and if the points outside the diagonal go in pairs: $p + iq$ maps to $q + ip$ and back, and conversely, $q + ip$ maps to $p + iq$ and back, the diagonal must contain at least one fixed point; in other words, the point "in excess with respect to parity" (i.e. the "+1" of $2k + 1$) must be on the diagonal. We have thus revealed at least one Goldbach decomposing prime of n.

Second case: the number of prime numbers between 3 and $n-3$ (both inclusive) is even; its square, which counts the number of points in the lattice, is even: we are not guaranteed to have a fixed point "all by itself". We then choose to add to the set of prime numbers $\mathcal{P}_n$ a single divisor of n. This has the effect of making the cardinality of $\mathcal{P}_n$ odd, and consequently its square odd as well, and we are brought back to the first case, why? Because a divisor of n can in no case be a Goldbach decomposing prime of n. This is expressed in the language of Gaussian congruences as follows:

$$(p \text{ divides } n) \wedge (p \text{ prime}) \iff p \text{ divides } n-p.$$

The divisor of n in question is associated with its symmetrical value with respect to the descending diagonal by a double back-and-forth between them (it is of the form $p+(n-p)i$ with $p \neq q$ and its "correspondent" is of the form $(n-p)+pi$.

We summarize the reasoning with the following diagram:



We have thus perhaps found a justification for the veracity of Goldbach's conjecture.

Appendix: Python program used for the experiments that corroborates the idea presented in the note

```
import math
from math import sqrt
import numpy as np
import matplotlib.pyplot as plt

def prime_sieve(N):
    is_prime = np.full(N, True)
    is_prime[:2] = False
    for p in range(2, math.isqrt(N) + 1):
        if is_prime[p]:
            is_prime[p*p::p] = False
    return np.nonzero(is_prime)[0]

class Primes():
    def __init__(self, N):
        self.__primes = prime_sieve(N)
    def __str__(self):
        return str(self.__primes)
    def __iter__(self):
        return iter(self.__primes)
    def __len__(self):
        return self.__primes.size
    def __getitem__(self, k):
        return self.__primes[k]
    def __contains__(self, x):
        k = self.index(x)
        return k < len(self) and self.__primes[k] == x
    def index(self, x):
        return np.searchsorted(self.__primes, x, side='left')
    def count(self, x):
        return np.searchsorted(self.__primes, x, side='right')
    def range(self, start, stop, step=1):
        return self.__primes[self.index(start):self.index(stop):step]
    def factors(self, n):
        if n in self:
            return np.array([n])
        else:
            P = self.range(2, n//2 + 1)
            return P[n % P == 0]

def syme(z,a,b):
    abar = a.real-a.imag*1j
    bbar = b.real-b.imag*1j
    zbar = z.real-z.imag*1j
    aaalpha = (a-b)/(abar-bbar)
    bbbeta = (b*abar-a*bbar)/(abar-bbar)
    return(aaalpha*zbar+bbbeta)
```

```
N = 104
P = Primes(N+1)
for n in range(6, 104, 2):
     trouve = False
     _,ax = plt.subplots(figsize=(10,10))
     milieu = n//2
     lf = P.factors(n)
     l1 = P.range(3, n - 3 + 1)
     l1 = np.array([x for x in l1 if x not in lf or x == milieu])
     l2 = n - l1
     listecomplexes = []
     for x in range(len(l1)):
          for y in range(len(l1)):
               listecomplexes.append(l1[x]+1j*(n-l1[y]))
     print('liste complexes = ',listecomplexes)
     print('diago')
     for z in listecomplexes:
          print(z,' image :  ',syme(z,0,n+n*1j))
     lg = np.array([x in P for x in l2])
     print(f' n = n (facteurs = lf) '.center(80, '_') + f'1 = l1'+ f'2 = l2')
     print('lg = [',end='')
     for k in range(len(lg)):
          if lg[k]:
               print(l1[k],' ',end='')
     print(']')
     for k in range(len(l1)):
          plt.plot([l1[k],l1[k]],[0,n],color='gray',alpha=0.8,zorder=0)
          plt.plot([0,n],[n-l1[k],n-l1[k]],color='gray',alpha=0.8,zorder=0)
     plt.plot([n,l1[-1]],[n,l1[-1]],color='gray',alpha=0.8,zorder=0)
     plt.plot([0,l1[0]],[0,l1[0]],color='gray',alpha=0.8,zorder=0)
     plt.plot([0,n,n,0,0],[0,0,n,n,0],color='gray',alpha=0.8,zorder=0)
     plt.plot(l1,l1,color='gray',alpha=0.8,zorder=0)
     for x in range(len(l1)):
          plt.annotate(l1[x],xy=(l1[x]-1,-2),fontsize=7)
     plt.scatter(l1[lg],l1[lg],marker='o',s=36,color='red',zorder=0)
     plt.title('n='+str(n))
     plt.axis('equal')
     plt.xlim(-2,n+2)
     plt.ylim(-5,n+2)
     plt.show()
     nomfic ='carreRQ'+str(n)
     plt.savefig(nomfic)
     plt.close()
```