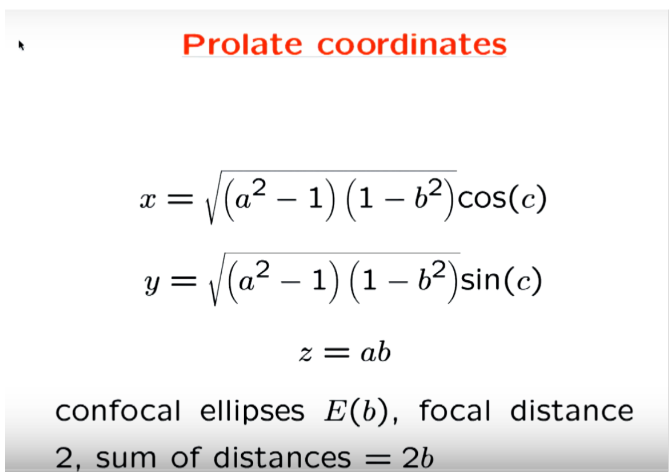
Prolate wave operator and infrared and ultraviolet for zeta

Alain Connes

Thank you so much for your invitation and for this occasion to participate to this great conference, which is a tribute to Christoph galensky who was a wonderful mathematician and physicist.

And what I want to explain is a quite surprising link between physics, when I talk about physics I mean I will talk about the prolate spheroid which is an ellipsoid but you consider it as a three-dimensional volume, I mean it's filled, and the zeros of the Riemann zeta function. And when we look at the zeros of the Riemann zeta function, when we plot them, what we find is of course a very strong similarity with what would be if you want the spectrum of a Dirac operator. And when you think about them spectrally, you find that there are two regimes which have to be understood if you want : there is the ultraviolet regime in which the counting function for the zeros of the Riemann zeta function is a very strange function ; it's given by a formula which is due to Riemann and which is this formula where you have, if you want, a term of the form E over 2 pi times log of E over 2 pi minus E over to Pi and then there is a logarithmic discrepancy which arises. So this is for the ultraviolet part. And for the infrared part, I mean, it's very surprising what you get because, I mean, like the first zero is around 14, and something and then okay, they behave in a very precise manner which has to be understood. And what we shall see is that both in the ultraviolet and the infrared, the prolate spheroid will play a crucial role.

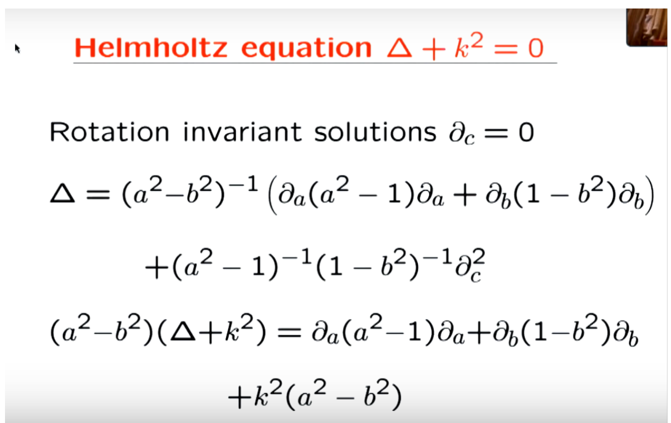


So I will talk about two joint papers very recent, one which was published in the Proceedings of the National Academy of Science with Henri Moscovici on the UV Prolate Spectrum and the second one which is in collaboration with Katia Consani and which considers this time the infrared behavior.

Okay so we shall see quite precise result, but somehow, let's start from the physics. Let's start from the prolate spheroid. So if you want, the Prolate spheroid is understood by means of a very specific system of coordinates which are called the prolate coordinates when you see them they look

Transcription de la vidéo derrière ce lien https://www.youtube-nocookie.com/embed/Yfu9$_fM - jzQ$, Denise Vella-Chemla, janvier 2023.

a little bit strange I mean there are these coordinates, a, b, c. c is an obvious coordinate because of the if you want the rotation invariance of the spheroid. But the other if you want the term which arises in front of the cosine and the sine for x and y looks rather strange. If you think a little bit, what you will find out is that in fact, this term is there because it describes a family of confocal ellipses so it looks like this : really what you have is there ; you have two foci ; you have one which is the point (zero, zero, one) and the other one which is the point (zero, zero, minus one) ; so they are both on the z-axis if you want, and the parameter b which was entering in the coordinates is in fact verifying that $2b$ is the sum of the distance into the foci. You know that an ellipse is defined by the fact that the sum of the distances to the two foci is constant and this constant is $2b$. So this is the role of the parameter b and the role of the parameter a is just an angular parameter. So what happens then is the following : what happens is that when you compute the Laplacian in this coordinates, the ordinal Laplacian okay, you find an expression which then will allow you to deal with the Helmholtz equation for the three-dimensional spheroid, the prolate spheroid (it's called prolate because it's elongated in the direction of the symmetry axis).



Okay so I mean when you look at the Laplacian, it has this form, okay, and if you look at rotation invariant solutions (things simplified a bit because you have the differential with respect to the variable c, the angular variable c which is zero), and then, what you get when you write down the Helmholtz equation, you find that there is what is called the separation of variables ; namely the equation splits if you want as a sum of two terms, one which only involves the variable a, and the other which only involves the variable b. And when you look a little bit more closely, what you find out is that in fact to solve the Helmholtz equation, what do you have to do ? You have to solve both the angular equation and the radial equation. So as equations, as differential equations, they are the same, but the variables that you use as variables a and b and the variable a is between -1 and 1. This is if you want the angular part. And the variable b is larger than one, it goes to infinity, and this is the radial part. Now for the solutions what the solutions will look like, the product of two functions one function which only depends on a and one function which only depends on b so $\phi(a)$, $\psi(b)$ but they have to be eigenvectors if you want for the same eigenvalue for the differential equation. And it is because they will be for the same eigenvalue if you want that the terms will cancel off and they will give you a solution of the Helmholtz equation.

Separation of variables

Angular equation = Radial equation, but the variables have domain $[-1, 1]$ for the angular part and $[1, \infty)$ for the radial part. Solution $\phi(a)\psi(b)$ with same eigenvalue and $\psi = 0$ on boundary.

Moreover of course, since you want to have Dirichlet boundary condition you have to have that $\psi(b)$ is 0.

So what happens, in fact, can be stated like this if you want is that there is an operator, there is a second order a differential operator which spits out from this Laplacian and which has the following form modulo easy rescaling which is a change of variables, the operator, the way we shall consider it, is this operator W_λ which depends on the parameter λ ; λ is related to K by a very simple equation k equals 2 Pi Lambda square, and the operator W_λ looks like this : it's a differentiation times lambda square minus x square times differentiation plus an additional term which is 2 pi Lambda x square okay.

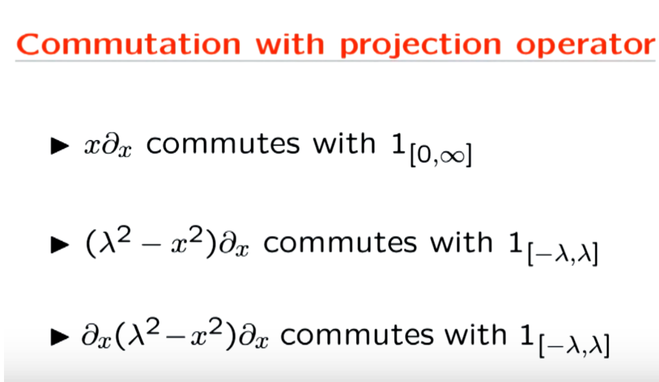
Prolate spheroidal operator

The second order operator $\mathbf{W}_\lambda$ appears from separation of variables in the Laplacian Δ for the prolate spheroid :

$$\mathbf{W}_\lambda := -\partial_x((\lambda^2 - x^2)\partial_x) + (2\pi\lambda x)^2$$

$$(k = 2\pi\lambda^2)$$

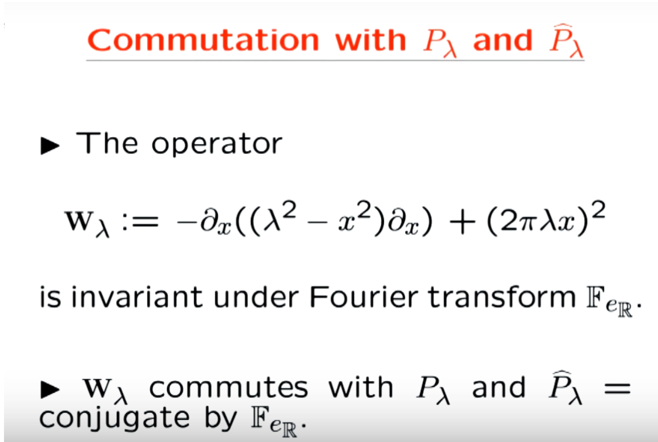
Now, this operator was used in a quite remarkable way by Slepian and his collaborators, who were working in Bell Labs and I will come back to their motivation later when we shall deal with the infrared problem. But somehow their essential discovery which is described in several papers so their essential discovery is the following : it is that in fact this differential operator commutes with what is called the truncated Fourier transform. And to understand this fact, I mean it's a little bit surprising because what you have to understand somehow is that these differential operators one I was showing before the W_λ in fact commutes with a projection operator. So this looks very strange because you know the projection operator on the interval [minus Lambda, Lambda], it's a function which is discontinuous I mean it takes a value 1 between minus Lambda and Lambda and takes the value of 0 elsewhere so I mean at first it looks very strange that the differential operator could commute with such a function, but if you think a little bit, you'll find out that in fact, it's not so surprising because the simplest case to consider would be only the operator xd by dx and to understand that it commutes with the projection on functions which have support on the positive integers.



Now why is this obvious ?

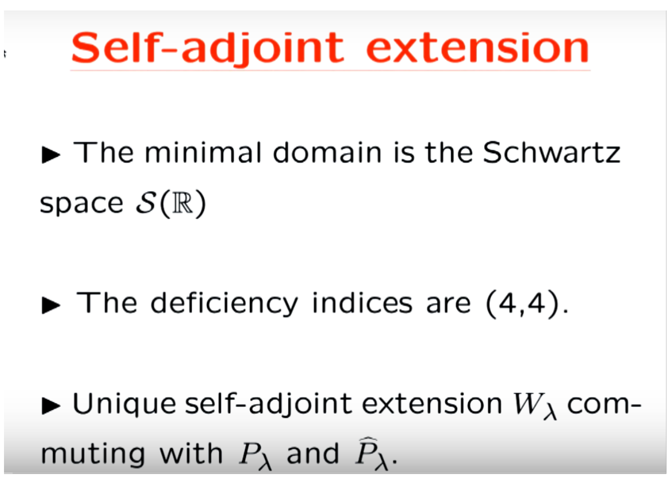
It's obvious because, if you want, the operator xd by dx generates the semi-group of the group actually of scaling operators, the group which replaces a function f of x by f of lambda x and so, I mean obviously you know, for lambda positive of course. So obviously it preserves the functions which have support in zero, infinity and it commutes with this projection.

So in fact, what happens is that the first piece of the operator which is d by dx lambda square minus x square times d by dx commutes with this projection on minus lambda, lambda and of course, the next term which was 2 pi lambda square times x square commutes of course with the multiplication by a function. So what happens is that not only this operator commutes with P_λ but in fact direct computation shows you that it commutes with Fourier transform, that's not difficult to see I mean, there is a piece of the operator which is the harmonic oscillator, which commutes with Fourier transform. So it commutes with Fourier transform, and because it commutes with Fourier transform *and* with the projection P_λ, it also commutes with the Fourier transform of P_λ. Namely, if you want, it commutes with the operator $\widehat{P_\lambda}$, which is obtained by conjugating P_λ by the Fourier transform. So that's what Slepian and his collaborators discovered, and I mean in the 90s when I got quite interested if you want in the zeroes of zeta, what I had found, I had used this projection P_λ and $\widehat{P_\lambda}$ to make a cutoff. And in fact, in my class in 98, I had been addressing the problem of treating the operator W_λ which normally is only treated in the interval minus lambda, lambda where it is self-adjoint as easily seen, to treat it on the full real line.

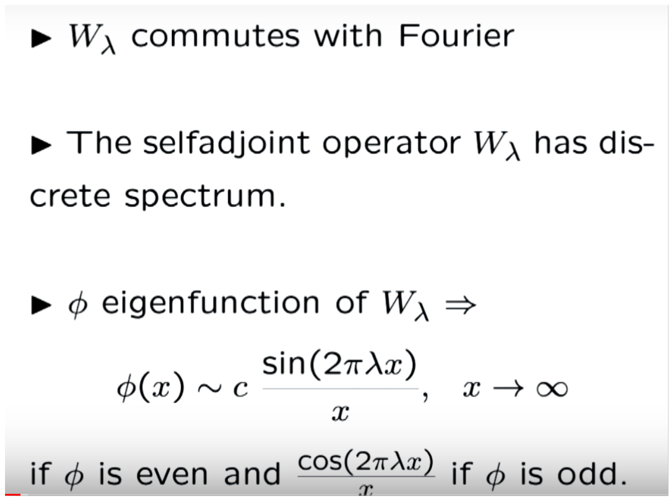


And when one treats it on the full real line... So what I had found at that time, I mean 98, was that if you take for the minimal domain of this operator the Schwartz space of Schwartz functions which have rapid decay as well as all their derivatives, then you find out that the operator is symmetric of course, but it's not self-adjoint, and in fact it has deficiency indices in the sense of von Neumann, which are both equal to four. And it has in fact a unique self-adjoint extension W_λ which is requested to commute with the projection P_λ and $\widehat{P_\lambda}$. That's where I stopped in 98, and that's where we started, two years ago with Henri Moscovici, we started our collaboration. And we did something which I had not dared to do many years ago in 98, mainly we really looked at the operator W_λ spectrally.

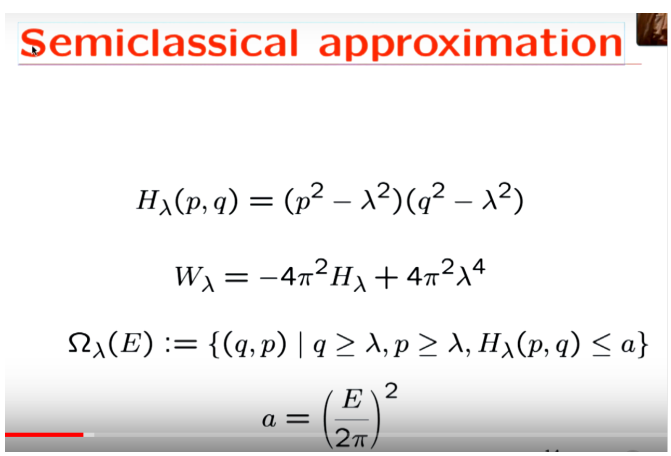
And what I will explain now is that if you want when you look at this operator W_λ spectrally, what you find out ? Well, of course, you find that it commutes with Fourier but this is this is sort of almost built in but to our great surprise with Henri, what we found is that the self-adjoint operator W_λ, on the full line now, not in the interval minus lambda, Lambda has a discrete spectrum.



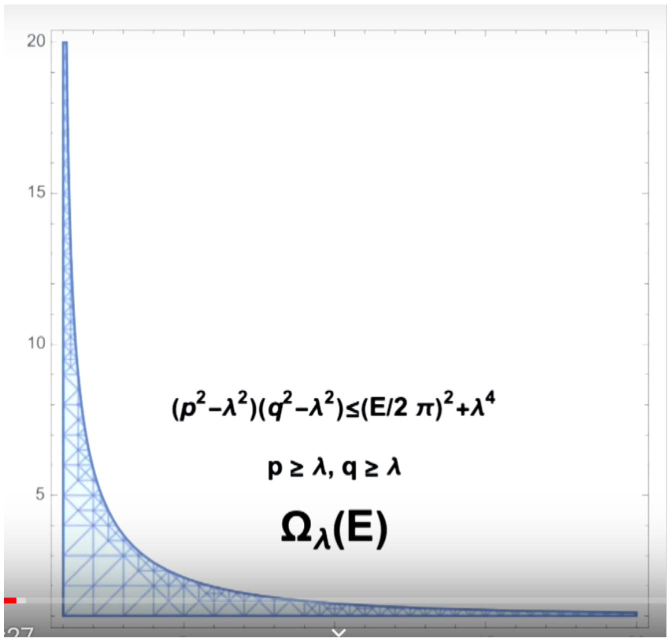
And what I will explain later is that this discrete spectrum will fit perfectly the ultraviolet behavior of zeroes of zeta. And what happens is that apparently nobody had looked at this spectrum, because this spectrum turns out to have both a positive part, and a negative part. And the reason why people were only interested in the positive part of the spectrum is that they wanted to fit with the separation of variables and they wanted to fit with the spectrum, which was corresponding to the finite interval and which is positive by construction. So nobody looked at the negative piece of the spectrum, and as we shall see, I mean, very roughly, the positive spectrum will correspond to the trivial zeroes of zeta and the negative spectrum will correspond now to the non trivial zeroes and to the ultraviolet behaviour of zeroes, it doesn't give exactly positions of zeroes but it gives the ultraviolet behaviour.



So I mean to find the self-adjoint extension, one has to give a boundary condition at infinity, and the boundary condition at infinity is in fact the request, if you want, that when you look at the even part of the spectrum, so the even function, which is the one we shall restrict to, then the function has to behave like a sine, it has to have this oscillatory behaviour of having many zeros of course, as you approach infinity, and to be equivalent to sine two pi lambda x over x when x goes to infinity. In the odd case, you have to replace the sine by a cosine.



Okay. So if you want to understand the spectrum of this operator as a physicist instead, it's very natural to look at the semi-classical approximation, and to express the operator W_λ in terms of an Hamiltonian, which you write classically because, okay, at this point, you don't care about non-commutativity of p and q ; so what you write is that W_λ is in fact expressed as minus 4 pi square times H_λ where H_λ is written above, it's p square minus lambda square times q square minus omega square, plus a constant which is not important. And then, when you want to look at the behavior of the spectrum, you have to compute an area and by computing this area, if you want, the area which is bounded by the value of H_λ, by computing this area, in fact, you will be able to estimate the number of eigenvalues of the operator. And when you do that, okay, you find this picture :



So you find a picture where, if you want, the thing which looks like an hyperbola is defined by the way in p square minus lambda square times q square minus lambda square equals E over to pi square plus lambda four, okay, and plus, we have these two conditions that p is larger than lambda and q is larger than lambda ; this is because of the positivity of the operator. Okay so that's what you have, and you make the computation

The area $\sigma(E)$ of $\Omega_\lambda(E)$ is given, with $a=\left(\frac{E}{2\pi}\right)^2$, by the convergent integral

$$I_\lambda(a)=\int_\lambda^\infty\left(\frac{\sqrt{a+\lambda^2x^2-\lambda^4}}{\sqrt{x^2-\lambda^2}}-\lambda\right)dx$$

$$\sigma(E)\sim\frac{E}{2\pi}\left(\log\left(\frac{E}{2\pi}\right)-1+\log(4)-2\log(\lambda)\right)+\lambda^2+o(1)$$

of the area okay I solved a difficult computation so you get an integral from lambda to infinity of a certain square root and so on, and what you find is that this integral in fact is given by elliptic integrals not elliptic functions, but elliptic integrals, in the sense of Legendre. And then, when you expand them with the correct idea I mean the idea that the operator in question will not be the Dirac operator but that it will be like the Laplacian, so you deal with a square of the Dirac operator, so you deal with that, and then, you find amazingly that the formula that you get for the number of eigenvalues begins to resemble extremely strongly the Riemann formula. It's E over 2 Pi times log of E over 2 Pi minus a term of order one which you would like to be minus one, so you would have to fit the lambda, so that it's minus one, and then plus lambda square plus a little o(1).

Okay. On the other hand, one knows that when making a semi-classical approximation, one has to be quite aware that it will give you a first idea but that you have to work much more in order to really give an estimate, and one can see that you know it couldn't really be completely right this way because you don't get the term in capital O of log E.

So in fact one has to push the analysis much further

In fact one has

$$I_\lambda(a) = \lambda^2 I_1(a\,\lambda^{-4})$$

and in terms of elliptic integrals

$$I_1(a) = aK(1-a) - E(1-a) + 1$$

$$\sim \frac{1}{2}\sqrt{a}(\log(a) - 2 + 2\log(4)) + 1 + o(1)$$

and okay so this is easy

Liouville transform

$$V(f)(y) := \Lambda^{1/2} f(\Lambda \cosh(y)) \sinh(y)^{1/2}$$

The operator V is a unitary isomorphism $V : L^2([\Lambda,\infty) \to L^2([0,\infty))$ which conjugates the operator W with the operator

$$S(\phi)(y) := \partial_y^2 \phi(y) - Q(y)\phi(y)$$

$$Q(y) = -(2\pi\Lambda^2)^2 \cosh(y)^2 - \frac{1}{4}\left(\coth^2(y) - 2\right)$$

So what one has to do, first of all, one has to do a Liouville transform to transform the operator, this prolate operator into if you want the usual Sturm-Liouville form and when you do that, okay, you get a potential, you get a potential which is very tricky because it's neither positive nor negative. You get this strange potential $Q(y)$ and then

Hamiltonian $H = p^2 + Q(q)$

(i) The Hamiltonian $H = -S$ is in the limit circle case at ∞.
(ii) The Hamiltonian H is in the limit circle case at 0.
Case $\Lambda = \sqrt{2}$ we get for the function $h = -Q$

$$h(y) = 16\pi^2 \cosh^2(y) + \frac{1}{4}\left(\coth^2(y) - 2\right)$$

M. Nursultanov, G. Rozenblum, *Eigenvalue asymptotics for the Sturm-Liouville operator with potential having a strong local negative singularity*. Opuscula Mathematica 37(1) :109

you have to apply so you have this hamiltonian now which is you know of strength p square because you're putting it in this form, okay, and it has this potential, and then one has to apply quite tricky estimates on the eigenvalue asymptotics from the Sturm-Liouville operator. Well, for the potential which had, if you want, a singularity.
So I mean these estimates exist, you do the calculations, and when you do the calculation, what you obtain,

Eigenvalue asymptotics for the Sturm-Liouville operator... 113

$h(p(\mu)) = \mu$, are

$$N(H,(0,\lambda)) = \pi^{-1}\int_0^\infty [(\lambda + h(x))^{\frac{1}{2}} - h(x)^{\frac{1}{2}}]dx + O(1),\ \lambda > 0, \qquad (1.4)$$

$$N(H,(-\mu,0)) = \pi^{-1}\int_0^{p(\mu)} h(x)^{\frac{1}{2}}dx + \pi^{-1}\int_{p(\mu)}^\infty [h(x)^{\frac{1}{2}} - (h(x)-\mu)^{\frac{1}{2}}]dx + O(1). \qquad (1.5)$$

so there are formulas if you want for each variable, available to compute this number of eigenvalues. And when you do that,

Formula for $N(a)$

$$N(a) = \frac{1}{\pi}\int_0^\infty \left((a + h(y))^{1/2} - h(y)^{1/2}\right) dy$$

At the level of the Dirac operator one has $a = (E/2)^2$

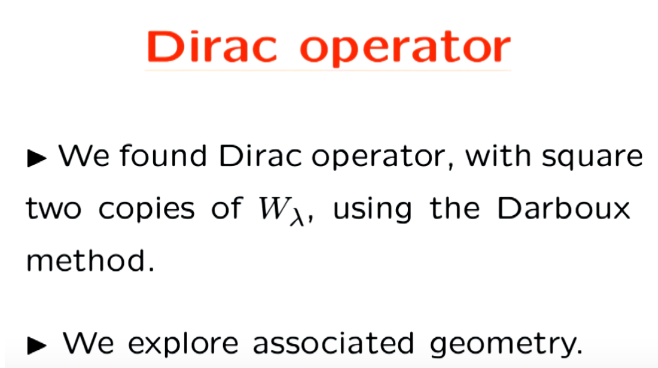
$$N_D(E) = \frac{E}{2\pi}\log\frac{E}{2\pi} - \frac{E}{2\pi} + O(\log(E))$$

The logarithmic term is $-\frac{1}{2\pi}\log E$. The numerical value of the coefficient is 0.159155 which is of the same order as the constant involved in the estimate of Trudgian for Zeta

$$|N_\zeta(E) - \left(\frac{E}{2\pi}\log\frac{E}{2\pi} - \frac{E}{2\pi}\right)| \le 0.112\log(E) + O(\log\log E)$$

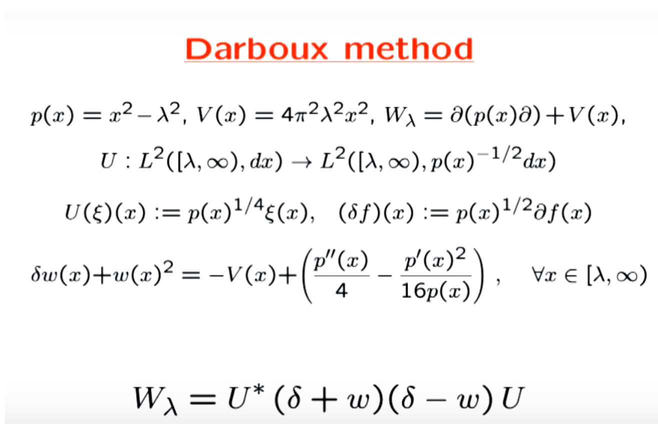
what you find now is much more complicated integrals. However, the amazing fact is that you find the correct E over 2 Pi log E over 2 Pi minus E over to Pi but you find that there is an additional term in Big O of log E as you would expect. And when you compute it you find that the relative term is in fact -1 over 2 pi times log E. And when you look at the numerical value of this coefficient you find that it is of the same order as a constant which is the one in the Riemann zeta function for the difference between the Riemann formula and something of ???[1].

So there is this very tantalizing fact and then of course, you know, this is not enough because I was taking the Laplacian



so we had to find the Dirac operator, we had to find the kind of square root, if you want, the kind of Dirac squareroot of this prolate spheroidal operator. And then, we wanted to explore the associated geometry.

So, how did we find the Dirac operator of the square root, if you want, of this prolate spheroidal operator. Well, we did it using the Darboux method.



The Darboux method is a very general method which is quite old and which allows you when you have a second order operator to write it not as a square but as a product of something of the form the operator of order one plus W and the operator of order one, the same operator of order one minus W. So one can do that, but in order to do that, you have to solve a Ricatti equation. So we have to find a solution of a non-linear equation which is delta W plus W square equals a certain

[1] $o(1)$?

function of the potential which is given which is given to you from the start. And this can be done in our case, because what one has to do, one has to find solution of the prolate operator which don't vanish.

Solution of Riccati equation

For $z \in \mathbb{C}$ and $u = u_1 + zu_2$ the solution u has no zero in (λ, ∞) if $z \notin \mathbb{R}$ and an infinity of zeros otherwise.

Solutions of the Riccati equation

$$w_z(x) = \frac{(x^2-\lambda^2)^{1/4}\partial\left((x^2-\lambda^2)^{1/4}u(x)\right)}{u(x)}$$

where $u = u_1 + zu_2$ and $z \in \mathbb{C} \setminus \mathbb{R}$.

Now of course as I showed you I mean the ordinary solutions vanish but when you combine two independent solutions with complex coefficients, then you find out that it vanishes nowhere.

So when it vanishes nowhere, what you can do is solve the Ricatti equation by a kind of logarithmic derivative which I have written as this $w_z(x)$ and for each complex number which is not in the reals, you find a solution of the Ricatti equation you find the corresponding Dirac operator.

Dirac operator

$$D = \begin{pmatrix} 0 & \delta + w(x) \\ \delta - w(x) & 0 \end{pmatrix}$$

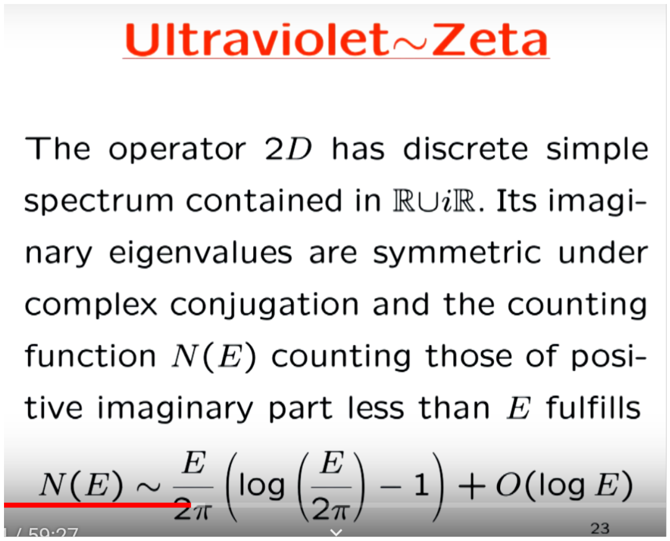
Then the square of D is diagonal with each diagonal term spectrally equivalent to W_λ,

$$U^* D^2 U = \begin{pmatrix} W_\lambda & 0 \\ 0 & W_\lambda + 2\delta w(x) \end{pmatrix}$$

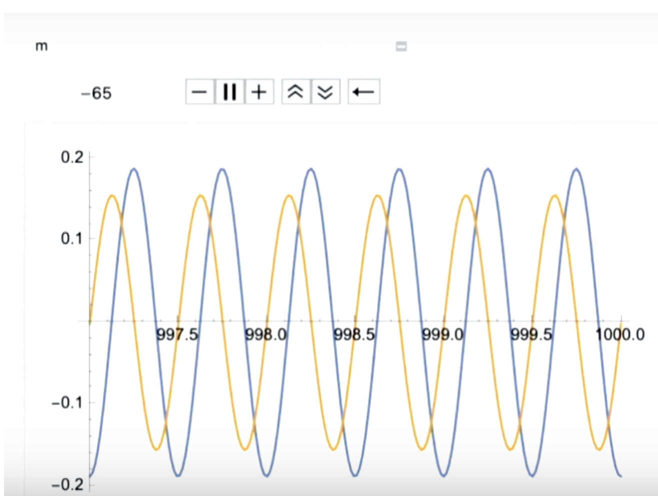
And all these Dirac operators are isospectral which means that you don't care which one you choose as far as the spectrum is concerned. So I mean, this is what we found, we found the Dirac operator, when you square it, it's of course a two by two matrix because it's acting something like spinors, okay, so it's a two by two matrix and when you square it, you find two copies of the prolate operator, namely first the prolate operator itself and then something which is isospectral to the prolate operator, okay (which it differs by a pair of prolate operators).

So that's what we found, and then if you want, by the previous computations because of course this Dirac operator having the square which is two copies of the prolate operator, you can compute its spectrum you can compute the number of eigenvalues and then we find, for that one, we find exactly the correct estimate for the number of eigenvalues.

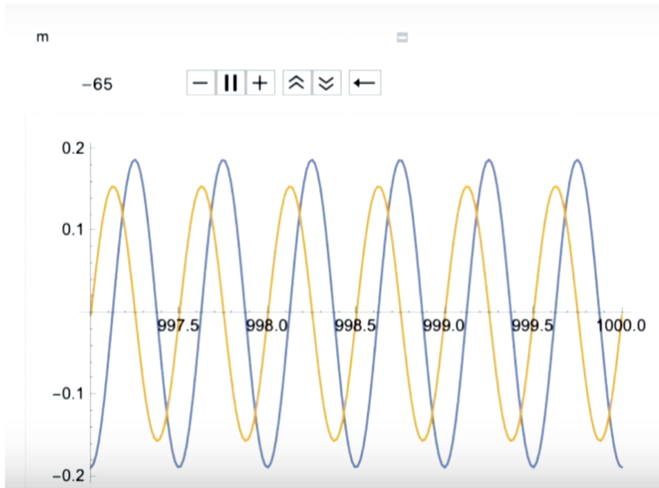
And I will show you very shortly how you do concrete computations and you compare with zeroes of zeta.

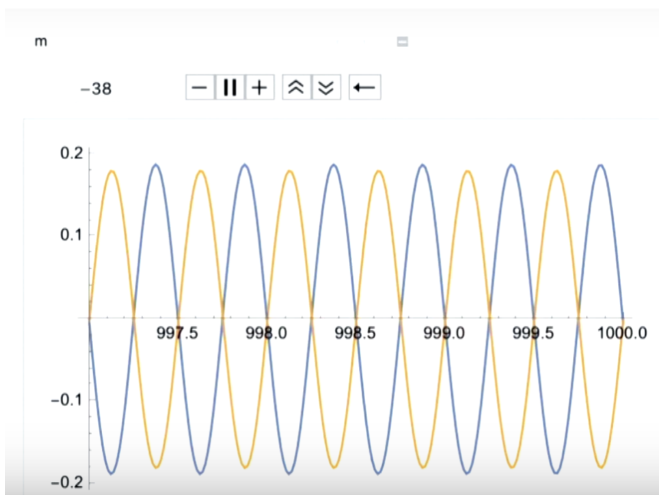


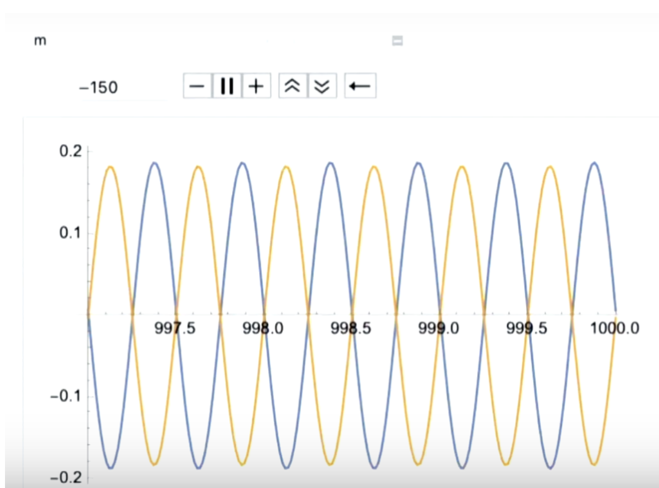
But how is this done okay ? This is done by, if you want, computing the negative eigenvalues of the prolate operator, and to compute these eigenvalues what you do is if you want, you expand the solution which satisfies a boundary condition at lambda, and you expand the solution for the eigenvalue which is minus 65 and you expand the solution which satisfies the boundary condition at infinity. And you try to match them, of course here they don't match, but when you move if you want the value minus 65, when you keep going, okay, you can see for instance at -38, they match except that there is a change of sign but of course this is not seen because you can always multiply one by minus one, so this is an eigenvalue minus 38 for the W_λ and the fact that they have opposite signs tells you that the function will be its opposite when you Fourier transform.



Now for the value minus 93, you can see that there is exact coincidence and this time, the function will be its own Fourier transform. So you keep going like that keep going like that, so this is minus 1 hundred 50, the next one, and what you see with the computer is that when you move the eigenvalue like around minus 150, the two sinusoids, if you want, the two oscillating pieces, they move with respect to each other. So after a while they coincide or they are strictly opposite to each other.







The first approximate negative eigenvalues of W are

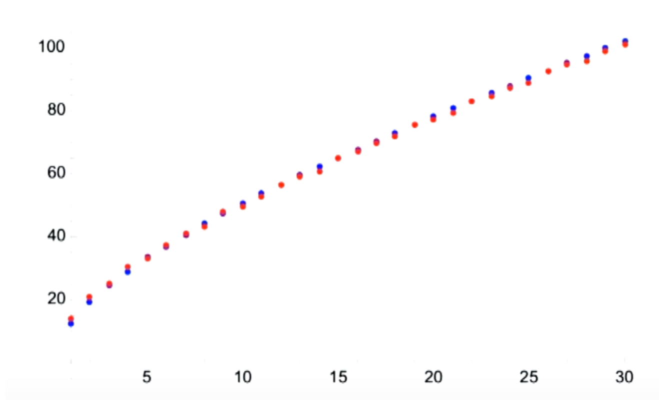
−39, −94, −152, −211, −279, −342, −416, −489, −561, −639, −718, −800, −887, −971,

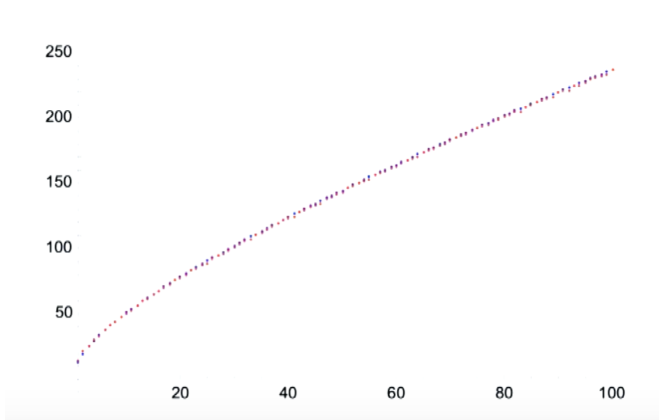
−1058, −1148, −1242, −1337, −1433, −1528, −1627, −1728, −1834, −1940, −2044, −2155, −2262, −2375, −2491, −2606, −2723, −2842, −2964, −3084, −3205, −3330, −3461, −3586, −3716, −3845, −3977, −4112, −4245, −4381, −4523, −4662, −4803, −4943, −5088, −5232, −5382, −5527, −5677, −5823, −5977, −6129, −6287, −6440, −6600, −6753, −6915, −7075, −7240, −7402, −7562, −7730, −7902, −8064, −8237, −8408, −8581, −8748, −8924, −9100, −9278, −9456, −9638, −9816, −10000, −10179, −10363, −10549, −10734, −10923, −11114, −11299, −11491, −11681, −11876, −12066, −12267, −12459, −12660, −12860, −13059, −13254, −13464, −13660, −13865, −14069, −14279, −14484, −14694, −14900, −15113, −15326, −15543, −15753, −15967

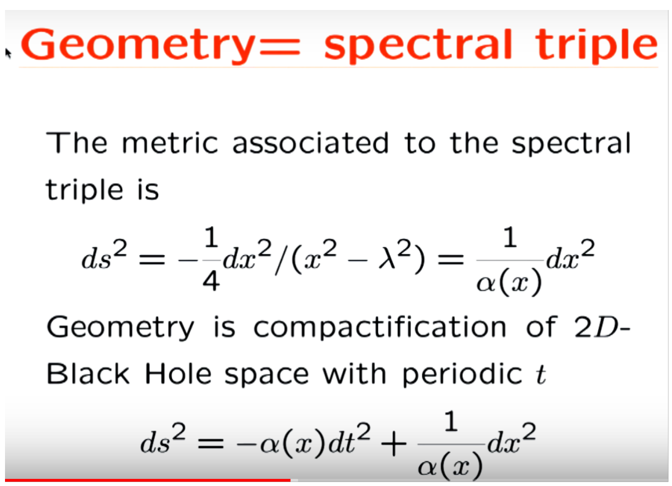
The comparison of $2\sqrt{-z}$ with the zeros of zeta then gives

102.098	101.318
104.365	103.726
106.621	105.447
108.885	107.169
111.068	111.03
113.225	111.875
115.412	114.32
117.661	116.227
119.766	118.791
121.918	121.37
124.016	122.947
126.127	124.257
128.25	127.517
130.307	129.579
132.378	131.088
134.507	133.498
136.558	134.757
138.607	138.116
140.613	139.736
142.66	141.124
144.665	143.112
146.724	146.001
148.688	147.423
150.692	150.054
152.617	150.925
154.622	153.025
156.576	156.113
158.581	157.598
160.499	158.85
162.481	161.189

So you compute like this the first approximate negative eigenvalues of W so you plot them, okay this is a lot of work, and now, okay, you pass to the corresponding Dirac operator and you compare it with the zeros of zeta. So you see, the first one is 12.49 instead of 14, yeah, and you keep going like that okay so you compare them, you compare them, and each time, the n-th one is very close to the n-th zero, it's not, you know, that time, changing the elements, no, the n is the same, okay, so keep going quite far, and when you plot them, so this is a plot where if you want the red spot represent the zeros of zeta and the blue spots that presents what we get spectrally and when you see only one spot it means that it's hiding so, it means, you know, that they really are pretty close to each other.







So we kept going up to the 60s first eigenvalues, up to the 100s eigenvalues and so on. And of course, so this really says that this self-adjoint operator, W_λ, because I mean remember okay I mean you know it's an extremely tantalizing hint that there should be an operator that will actually exactly deal with them. The next step of course would be to deal with the other part I mean with precise values and infrared part of the spectrum. But before I do that, what is very quite important in non-commutative geometry is that geometry is given spectrally. When you speak about a geometry, you can define it by means of the Dirac operator, what replaces the Dirac operator, and what replaces of course the functions, and so on, so here, it's not so difficult to find which functions are and, I mean because they are like you know functions on this half line and to look at the metric

: the metric is obtained from the symbol of the Dirac operator and here, the metric that you find is given by a dx square, there is one force over x square minus lambda square. But what is quite amazing is that when you look at the spectrum as I said before, so there are negative eigenvalues for the Laplacian but there are also positive eigenvalues. So this means that when you pass to the Dirac operator, it will have purely imaginary eigenvalues like for the zeroes of zeta, but it will also have real eigenvalues. And what we checked, with Henri, is that you know the these real eigenvalues, they have exactly the same behavior as a trivial zeroes of zeta. But the fact that you have both things, namely the negative and the positive eigenvalues tells you that you are not dealing with a Riemanian problem, you are dealing with a Lorentz problem, I mean with a Minkowski-type metric. And so, I mean we used in order to have a first idea of the geometry which is behind it, we used to view it, simply, as a compactification of a two-dimensional Lorentzian geometry. And okay it turns out that, you know, there is a way to do that, where you make the variable t-periodic, and this corresponds to a black hole in two dimensions.

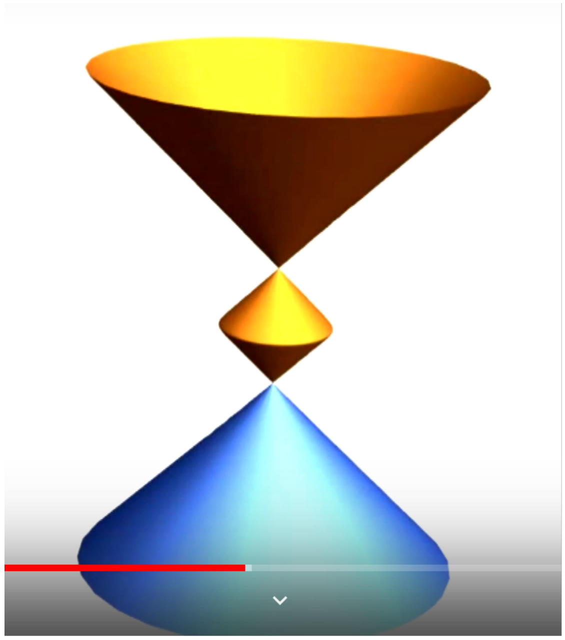
which after changing coordinates to $v = t - t(x)$ with

$$t(x) = \frac{1}{8\lambda} \log\left((\lambda + x)/(x - \lambda)\right)$$

becomes smooth (black hole trick)

$$ds^2 = 4\left(x^2 - \lambda^2\right) dv^2 - 2dvdx$$

Okay and I mean, you can do the usual trick, which is to see that the metric of the black hole is in fact smooth. So you can really rewrite the metric by a suitable change of variables to make it smooth. And I mean there is, if you want, a way to embed this in Minkowski-three space, and when you embed in Minkowski-3-space, what is quite interesting is that in order to construct the embedding, you have to use again elliptic integrals. And it will give you this picture



so if you want the the part which is above which is looking like a cone and corresponds to the interesting piece of the prolate operator. The part which is in between corresponds to what happens on the interval, and the part which is below, okay, is of course the symmetric of the upper part. And when you look at the geodesics, they have the usual stuff the suspected.
